Decision Modeling

David M. Tulett

Version 1.1

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https://linney.mun.ca/pages/view.php?ref=36808

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Version History

1. Preliminary versions were developed over the period 2011-2018 for use in the Faculty of Business Administration at Memorial University of Newfoundland.

2. Version 1. The first version advertised through INFORMS and CORS was made in June 2018. Minor changes were made as follows:

   (a) Version 1.0.1. November 14, 2018.

   (b) Version 1.0.2. November 29, 2018.

   (c) Version 1.0.3. February 5, 2019.

   (d) Version 1.0.4. March 1, 2019.

3. Version 1.1. Content on doing sensitivity analysis using LINGO was added to Appendix A, the front-end of this document was re-designed, and some minor changes were made. This revision, compiled on July 24, 2019, is the current version.

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Acknowledgements

1. This document was written using the very versatile LaTeX and PSTricks programs, which are open-source software. LaTeX is particulary good at

writing mathematical expressions, and PSTricks produces excellent graphics. Both can be downloaded for free from the TEX Users Group (TUG) at http://www.tug.org.

2. A feature of Excel that is heavily used in this document is the Solver. The Solver is made by Frontline Systems, Inc., https://www.solver.com.

3. The author acknowledges suggestions and comments from many students and colleagues based on earlier editions of this document, which began in 2011.

4. The author wishes to thank retired professor Austin Redlack, Ph.D, with whom he worked on materials for courses that were developed in the period 1987-1997, and which continued to be used in courses that were offered up to 2011. In a few cases, examples from those works have been maintained in this document.

5. Since June of 2018 this document has been hosted on Memorial's Linney. System. The author wishes to thank Ms Vanessa Mackey of Memorial University's Centre for Innovation in Teaching and Learning (CITL) for uploading the numerous updates.

6. The author thanks his wife Mary for her proofreading and support for the writing of this document.

Author

David M. Tulett obtained a B.Sc. in mining engineering and a Ph.D in management (dual major in operational research and finance) from Queen's University (Canada). He is member of the Faculty of Business Administration at Memorial University, St. John's, NL, Canada.

Preface

This e-book was developed for Business 2400 at Memorial University of Newfoundland, Canada. In June of 2018, this document was made available to the entire world through Memorial's Linney system. From time to time changes to this document will be made. These versions are listed on page i. This version was made on July 24, 2019.

In this e-book, a word, phrase, or page number in red indicates a link to somewhere else in this document; a word or phrase in pink indicates a link to the web. In particular, both the Table of Contents and the Index are linked to the appropriate places in the main body of the material.

The reader needs to have learnt, or be prepared to learn, the basics of spreadsheets. A quick overview of the basics of spreadsheet operations using the syntax of Excel is provided in Chapter One. This e-book should be read in conjunction with any other materials which have been required for your course.

A great deal of care has gone into the preparation of this document, but there may well be some errors lurking somewhere. Readers who spot such errors, or who simply want to offer suggestions for improvement, are encouraged to contact the author by email.

Each of the nine chapters ends with a section called *Problems for Student Completion*. The solutions for these problems (nine pdf files) may be obtained from the author.

David M. Tulett

Email: dtulett@mun.ca

Contents

Chapter 1

Introduction

1.1 A Paradigm for Problem Solving

Decision Modeling involves the creation of mathematical models which represent problems faced by business management. To a lesser extent, it also involves numerically solving these models. When a mathematical topic is needed which requires more than first-year mathematics, it is extensively reviewed in this course. Often, the numerical calculations can be left to a spreadsheet or other software tools. If there's a difficulty with this subject, it's probably not the mathematics.

Instead, the difficulty is likely to be the building of the *model* which the mathematics seeks to solve. The important thing is always going from a problem description to a model for the problem. This is part of the paradigm of managerial problem solving by mathematical analysis, which can be thought of being composed of four phases:

1. problem definition
2. model building
3. solution
4. implementation.

When the fourth phase has been done, it is appropriate to ask whether or not it addressed the original problem. Because we are working in an academic context, we cannot observe the entire paradigm. The "problem" is not for us a real-world observation, but instead it is a written description (a "word" or "story" problem).

Also, we cannot implement the solutions. We are left with looking at the second and third phases of the paradigm. This book heavily emphasizes the second phase, though some simple solution methodologies are introduced. Most models fit into a general class of models for which solution software has been written and is widely available.

In the next section we will see a short example of the paradigm which evaluates mobile telephone plans, but first we go through the basic concepts of spreadsheets.

1.2 Spreadsheets

1.2.1 Introduction

Often, spreadsheets are a good way to solve numerical problems. All spreadsheets will have an array of rows labeled 1, 2, 3, and so on, and columns labeled A, B, C, and so on. The intersection of a particular row and column is called a cell, which is denoted by giving the column letter followed by the row number, e.g. B5. By putting numbers and formulas into the cells, we can perform all sorts of mathematical operations, ranging from simple addition, to optimization. Here is what an array going from cell A1 to cell F10 (also denoted as A1:F10) looks like:

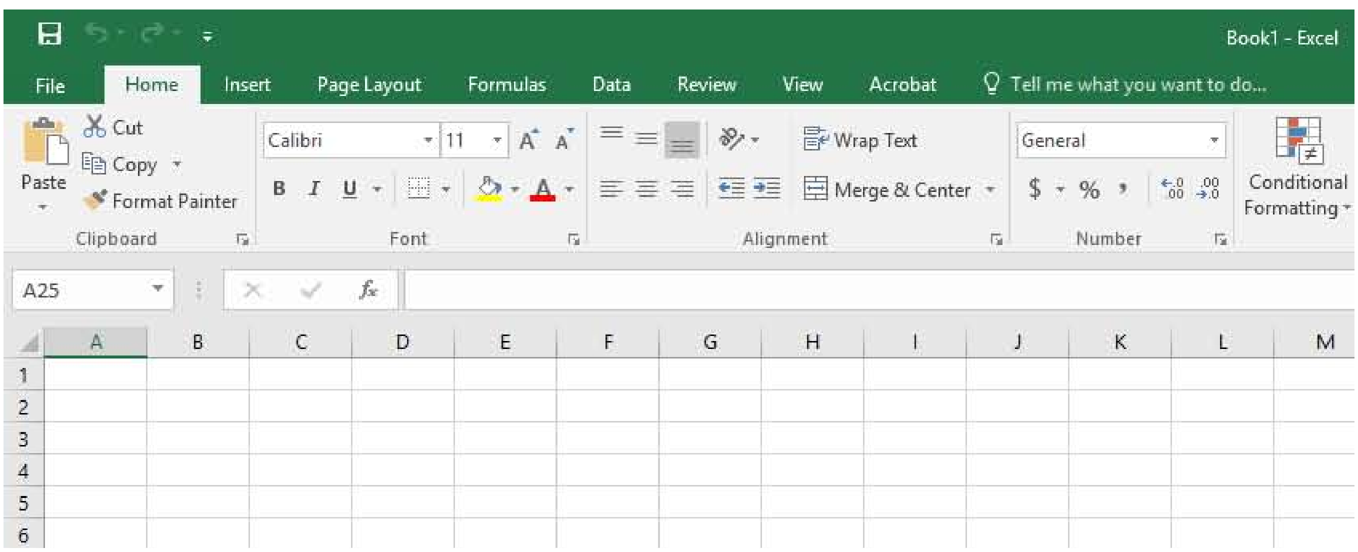
	A	B	C	D	E	F
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						

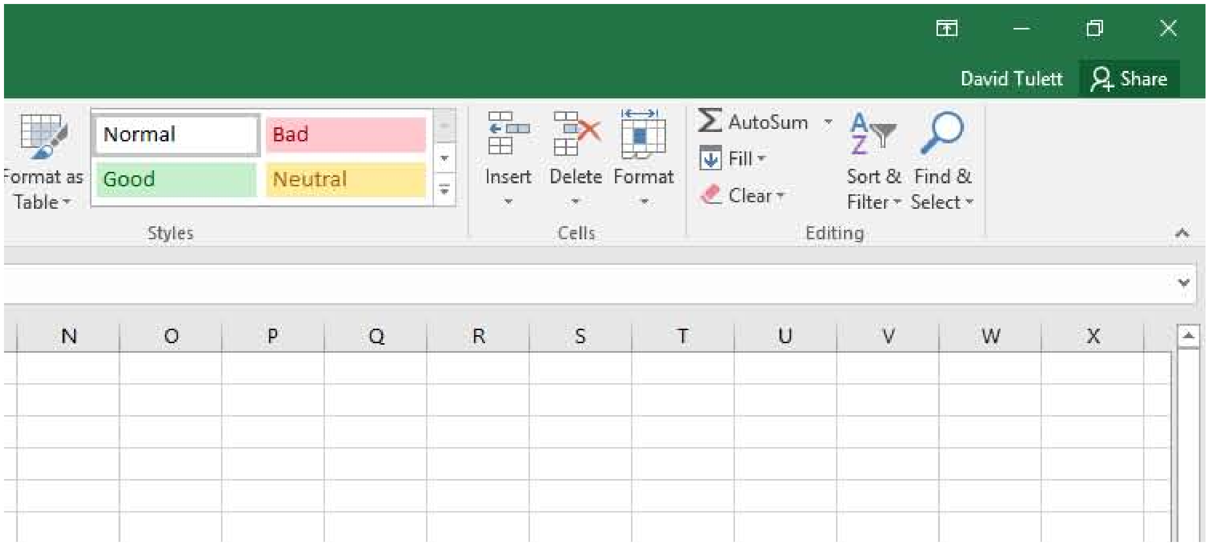
A spreadsheet package comes as part of a set of software programs. The dominant software is Microsoft Office 365 (for both Windows and Macintosh), but there are others, both non-free and free. A review of several free alternatives to Microsoft Office is available online.[1] Despite the free competition, the

[1] See https://www.techradar.com/news/the-best-free-office-software.

Excel spreadsheet program from Microsoft Office remains dominant. There are several reasons for this: product quality, a need for absolute file compatibility, and a need for features not found in some of the free programs. The creation of spreadsheets in this document was done using the Windows version of Excel from Microsoft Office 365. Readers who own the Apple Mac version will see some minor differences. In what follows, the instructions assume the Windows operating system. Users of a Macintosh should consult https://sway.office.com/TLT-kbLbBsVXZrX2.

What makes Microsoft Excel different from other spreadsheet packages is the command structure which appears above the row of column letters. Each of the tabs contains a large number of commands. Even for just the Home tab, there are so many commands that the screen shot has been split into two parts:





There is some commonality amongst all Office programs (Word, Access, etc.). For example, in the very top row there is the Quick Access Toolbar near the left, the Title bar in the centre, and the minimize/maximize/close buttons on the right. Also, on the left edge of the second row, the File tab is common to all Office programs. The File tab occupies the top-left position of what is called the *Ribbon*, the rectangular area between the Title bar and Formula bar (the Formula bar is just above the row of letters for the columns).

At the top of the Ribbon is the list of tabs. After the File tab, there are tabs named Home, Insert, Page Layout, Formulas, Data, Review, View, and Acrobat. To the right of the Acrobat tab help can be obtained by clicking on "Tell me what you want to do ...". The Home tab is the default when opening a file, and it is the tab used most often. When a tab is clicked, a whole new set of commands becomes available. There is only space here to consider a few of these. For more information, there are free on-line tutorials, such as https://edu.gcfglobal.org/en/excel2016/.

To begin using Excel, we consider the four basic arithmetical operations. As one would expect, we use $+$ for addition and $-$ for subtraction. Multiplication uses the symbol $*$, which seems strange at first but it avoids confusing the standard multiplication symbol $\times$ with the small letter x, or its capital, X. For the fourth operation, which is division, there's no $\div$ symbol on the keyboard, and so we use a forward slash / instead. For example, 2/5 means 2 divided by 5. These basic expressions can be used to create a formula, which in Excel must begin with the = symbol. For example, $3 + 2 \div 5$ would be entered as =3+2/5.

Powers are handled using the ˆ (caret) symbol.[2] For example, to find 2 multiplied by itself five times (2^5), we enter `=2^5`. In summary:

$+$	addition
$-$	subtraction
$*$	multiplication
$/$	division
ˆ	powers

Spreadsheets contain hundreds of built-in functions which can be used to help build formulas. They can be accessed from the Formulas tab, or simply be used directly on the main part of the spreadsheet if the user already knows the function and its syntax. Most users will memorize the function names for common tasks, such as summing a column of numbers, but will refer to the Formulas tab to access a function that is less familiar. Sometimes a formula only consists of the = symbol followed by the name of the function (denoted in this document using capitals) and its argument enclosed in brackets. Of course, a formula can also be complex using a combination of operators ($+$, $-$, and so on) and several functions.

As an example of a simple formula, to sum an array of numbers in say C3:C8, we use the **SUM** function, and the full writing of the formula is:

=SUM(C3:C8)

Normally, one would want the sum to appear in the same column below the other numbers, in cell C9 or C10, but it could be placed anywhere on the spreadsheet. The **AVERAGE**, **MIN**, and **MAX** functions are as follows. If we want the average of the numbers, this is =AVERAGE(C3:C8) as one might expect. The smallest number in the range is =MIN(C3:C8), and the largest is =MAX(C3:C8).

The default number formatting on Excel uses scientific notation to display very small or very large numbers. For example, it might calculate a number which theoretically should be zero, but due to a tiny bit of numerical error it is computed as -0.0000000000018. Excel will display this number as $-1.8\text{E}{-12}$. If desired, the formatting can be set to override the default setting.

1.2.2 Example – Calculating Students’ Final Marks

There are many students in a class, and the professor needs to find the average mark on each test/exam, and wishes to compute each student’s mark. There are

[2]The caret symbol, found above the number 6, is also called a *circumflex*. However, technically a circumflex is raised so that it can be put above something, such as in the French word *fête*.

two tests worth 25% of the final mark, and a final examination worth 50%. However, if and only if it helps the student, the final examination for that student will be worth 100%. The computation of the grade will compute a mark such as 68.7, which will have to be rounded to the nearest integer.

In order to save space, we will illustrate this example using just the data for six students. The raw (fictitious) data are:

	A	B	C	D	E	F
1	Name	Test 1	Test 2	Exam		
2						
3	Aylward, Susan	84	75	82		
4	Chang, Wi	62	69	63		
5	Murphy, Joseph	36	51	47		
6	Noonan, Anne	55	46	49		
7	Shawanda, Janet	76	81	77		
8	Wilson, John	92	88	89		
9						
10						

Column A gives a list of students ordered alphabetically by surname. In order to make each student's surname/given name fit the space, we had to make column A bigger than its default value. We can do this visually by using the mouse to drag the line between columns A and B to the right. While this visual method will change the column width for all the rows, we would have to scroll down through all rows to verify that we made the column wide enough. Instead of using this visual method, all we need do is double-click on the line between A and B. The double-clicking is a shortcut to replace going to the right-hand side of the Ribbon (in the Home tab), clicking on Format above the word Cells, and then under Cell Size, clicking on AutoFit Column Width.

Now we will add Raw Mark and Final Mark titles to columns E and F respectively. When inputting formulas, we try to minimize the number of times that we enter formulas by hand; we will use the spreadsheet's ability to copy cells as much as possible. Therefore, in row 3 of column E, we enter a formula which will be copied into the other cells in column E. The user can specify either absolute or relative cell referencing. In absolute cell referencing, a dollar sign in front of a column letter in a formula freezes the column, a dollar sign in front of the row number freezes the row, and dollar signs in front of both freezes both the row and the column. We will see an example of this later. Here, however, we want relative cell referencing. In relative cell referencing, moving a column to the right increments the column letter in a formula, and moving downwards increments the row number.

In cell E3, we need to calculate the student's mark based on the numbers in cells B3, C3, and D3. If the computation were to be based solely on a 25/25/50 apportionment, the formula would be easy. On a spreadsheet, multiplication uses an asterisk, so for example 25% of the number in cell B3 is 0.25*B3, and the total raw mark would be:

=0.25*B3+0.25*C3+0.5*D3

However, only the final exam will count if and only if it is to the student's advantage. Hence we want either the formula above, or the number in D3, whichever is higher. Using the MAX function the formula to be placed in cell E3 for the raw mark which reflects whichever method of computation is better is:

=MAX(0.25*B3+0.25*C3+0.5*D3,D3)

The **ROUND** function will round any number to a specified number of digits to the right of the decimal place. Since we want an integer, the specified number of digits is 0. In cell F3, we type =ROUND(E3,0). As an aside, we note that the INT command is not what we want here; it will always round down to the nearest integer. However, using =INT(E3+0.5) does do the same thing as =ROUND(E3,0).

We now need to copy cells E3 and F3 into rows 4 to 8 of columns E and F. The newer "drag and drop" method is:

1. Use the left button of the mouse to click on cell E3.

2. Keeping the left button down, drag the mouse over to cell F3, and then release the left button.

3. Use the mouse to place the cursor at the bottom-right hand corner of the range. It will change from a thick white cross to a thin black one.

4. Drag the cursor to the bottom-right hand corner of the range to be filled in (F8 in this example) and then release the button.

The traditional (but slower) way to do this is:

1. Use the left button of the mouse to click on cell E3.

2. Keeping the left button down, drag the mouse over to cell F3, and then release the left button.

3. Use the mouse to place the cursor at the bottom-right hand corner of the range. Click on the right button of the mouse. A vertical menu with the word Cut at the top and Hyperlink at the bottom will appear.

4. Move the mouse to the word Copy, and then click on the left button. A blinking edge will appear around the range E3:F3.

5. Click on the left button on E3, and drag across to F3 and down to F8, and release the left button. Cell E3 will be white; the rest of the range E2:F8 will be light grey.

6. Press the Enter key.

In cell E4 the number 64.25 will appear. Clicking on this cell gives the cell's formula in the Formula Bar just below the Ribbon. Here we see the formula =MAX(0.25*B4+0.25*C4+0.5*D4,D4). In copying cell E3 above it, the spreadsheet updated B3 to B4, C3 to C4, and D3 to D4.

In row 10, we write the word Average in column A, and then use the next five cells in row 10 for the averages of the numbers in columns B through F inclusive. The word Average in cell A10 is just a label; Excel doesn't use it as a command. For this, we need to use the built-in **AVERAGE** command. The average of the numbers in rows 3 to 8 inclusive of column B is =AVERAGE(B3:B8). This can be entered by typing this out, or by typing =AVERAGE(and using the mouse to click on cell B3 and dragging it down to cell B8, and then typing the right bracket.

Using the procedure described above, we copy the contents of cell B8 into the range B8:F8. We have done what we set out to accomplish, but we can also improve the visual appearance of the spreadsheet. In the graphic below, we have done the following on the Home tab:

1. The titles have been bolded. We click on the relevant cells, and then click on the **B** command located on the ribbon below the word Home.

2. Some of the cells have been given borders. The set of commands to create borders is located a few centimetres to the right of the **B** command.

3. The cells which contain the titles have been coloured yellow. The tilting paint can to the right of the Borders command can be clicked to reveal a

palette of colour choices. Yellow is one of the standard colours at the bottom.

	A	B	C	D	E	F
1	**Name**	**Test 1**	**Test 2**	**Exam**	**Raw**	**Final**
2					**Mark**	**Mark**
3	Aylward, Susan	84	75	82	82	82
4	Chang, Wi	62	69	63	64.25	64
5	Murphy, Joseph	36	51	47	47	47
6	Noonan, Anne	55	46	49	49.75	50
7	Shawanda, Janet	76	81	77	77.75	78
8	Wilson, John	92	88	89	89.5	90
9						
10	**Average**	67.5	68.3333	67.8333	68.375	68.5

The formula for the active cell is always visible in the Formula Bar, but sometimes we want to see all the formulas without having to see them one by one. This is accomplished by holding the ***Control*** key (labeled ***Ctrl*** at the bottom-left of the keyboard) down, and then clicking on the key below the Escape key that has a tilde (˜) on top and an single left quotation mark (‘) symbol underneath. Equivalently, this can be done under Formulas/Show Formulas. What we obtain by doing this is called *formula view*. Repeating this procedure brings the user back to the usual numeric display, called *normal view*.

In this example Columns E and F need to be widened to see the entire formulas. Doing this, they appear as:

	E	F
1	**Raw**	**Final**
2	**Mark**	**Mark**
3	=MAX(0.25*B3+0.25*C3+0.5*D3,D3)	=ROUND(E3,0)
4	=MAX(0.25*B4+0.25*C4+0.5*D4,D4)	=ROUND(E4,0)
5	=MAX(0.25*B5+0.25*C5+0.5*D5,D5)	=ROUND(E5,0)
6	=MAX(0.25*B6+0.25*C6+0.5*D6,D6)	=ROUND(E6,0)
7	=MAX(0.25*B7+0.25*C7+0.5*D7,D7)	=ROUND(E7,0)
8	=MAX(0.25*B8+0.25*C8+0.5*D8,D8)	=ROUND(E8,0)
9		
10	=AVERAGE(E3:E8)	=AVERAGE(F3:F8)

1.2.3 Putting Excel files into other Documents

One advantage of using Microsoft Office is that each of the tools, Excel, Word, PowerPoint, etc. work well together. If a Word document is being written, and if we want to imbed the spreadsheet used above, all we have to do is go to the Excel spreadsheet, click on cell A1 and drag the mouse to cell F10, press the Control key down, and then click on the key for letter C. We can now go to a Word document, press the Control key down, and then click on the key for letter V, and a picture of the spreadsheet will appear. There are some limitations of this click-and-paste method, however. The gridlines will not appear, and the row numbers and column letters will not appear.

At the other extreme, if we want to see everything as it appears on the screen, we can click on the Print Screen key. Doing this saves the image on the screen to the Clipboard. Going to Word and using Control V will make the picture appear in the Word document.

Sometimes we want an image which is in-between the two choices above, i.e. we want the main body of the spreadsheet but with the gridlines and with the row and column headings as well. With Adobe Acrobat or similar pdf-creator installed we can do this as follows:

1. Click on the File tab in Excel, and then click on Print.

2. Set the Printer to Adobe PDF.

3. Under Settings one usually wants Print Selection.

4. At the bottom click on Page Setup, and when the dialog box appears click on Sheet.

5. Under Print, click on the boxes for Gridlines and Row and Column Headings. Click on OK to close the dialog box.

6. When one clicks on the square labeled Print at the top, a pdf file is created, and the user is prompted for a filename.

Note that this procedure causes nothing to be sent to a physical printer. The pdf file can be imported into other documents, though cropping the image first using Adobe Acrobat or a similar product would probably be advisable.

1.2.4 Further Excel Functions

So far, we have seen five arithmetical operators ($+$, $-$, $*$, $/$, and ^) and the following functions: **AVERAGE**, **INT**, **MAX**, **MIN**, **ROUND**, and **SUM**. Here are some more functions which are quite important.

The IF function

An **IF** function has three parts to its argument. The first is a question, which Excel needs to evaluate to see if it's true or false. The formal name for such a question is a "logical test". If it's true, then Excel goes to the second part of the argument and follows the instruction given there; if it's false, then Excel goes to the third part of the argument and follows the instruction given there.

In the logical test we might want to know if the value in one particular cell is less than or equal to the value in another cell. Because there is no $\leq$ symbol on the computer keyboard, we would use <= instead. Similarly, >= means $\geq$.

Here's a simple example. We want there to be a 1 in cell B1 if the number in cell A1 is 30 or more, and for there to be a -1 in cell B1 otherwise. We make cell B1 active and type =IF(A1>=30,1,$-$1). If we type 5 into cell A1, we obtain -1 in cell B1. If we type 34 in cell A1 we obtain 1 in cell B1.[3]

The expressions in each of the three sections can be much more complicated than this simple example. Indeed, the second and third parts can contain the IF function, which is called a "nested IF". Nesting can be up to 64 levels deep.

The IFS function

In 2016, a new function named **IFS** was introduced into Excel which can replace a nested IF. An example which is solved using both the traditional nested IF function, and the new IFS function, appears on page 21.

The SUMPRODUCT function

We often need to find what is mathematically called the inner product of two rows, more commonly referred to as the "dot product". Suppose we buy 10 apples, 12 oranges, and 8 bananas, each of which cost \$0.55, \$0.50, and \$0.30 respectively. Hence we spend $10(0.55)+12(0.50)+8(0.30)$ = \$13.90 in total.

[3] If we type 29.9999999999 in cell A1, then depending on the formatting we might see 30 in cell A1, but B1 will be computed as -1 because the true value of 29.9999999999 is in the computer's memory.

On a spreadsheet we can put the per-unit prices into one row, and the per-unit quantities in another. Let us suppose that we put the former in the range C2:E2, and the latter in the range C3:E3. The brute-force way to find the total cost for any set of numbers would be to compute:

=C2*C3+D2*D3+E2*E3

With only three items, this works well enough, but it would be quite tiresome if we had say twenty items. This is where the **SUMPRODUCT** function is useful. It finds the dot product where the argument is range1,range2. For this example, the expression is:

=SUMPRODUCT(C2:E2,C3:E3)

Here this is shown on a spreadsheet in Formula view:

	A	B	C	D	E
1	Total Cost		Apples	Oranges	Bananas
2		Price/Unit	0.55	0.5	0.3
3	=SUMPRODUCT(C2:E2,C3:E3)	Quantity	10	12	8

Other Functions

Other Excel functions used in this book or needed for solving the end-of-chapter problems are as follows:

Absolute Value The **ABS** function is used to find the absolute value of its argument. For example, `=ABS(C2)` in cell D2, where the number in C2 is -230, puts 230 into cell D2.

Powers to the base ***e*** To find *e* (2.71828...) raised to an exponent we use the **EXP** function. For example to find e^2 we use `=EXP(2)` to obtain 7.389056099.

Square Root The **SQRT** function is used to find the positive square root of a number. For example, `=SQRT(B11)` in cell A7, where the number in cell B11 is 25, puts a 5 into cell A7. If the number in cell B11 is negative, an error message will appear in cell A7.

1.2.5 Excel Array Formulas

Array formulas in Excel allow for multiple calculations on one or more of the cells in an array. Array formulas require simultaneous use of the ***Control***, ***Shift***, and ***Enter*** keys.

Matrix Multiplication

A matrix is a rectangular array of numbers, with m rows and n columns. The product of two matrices is defined if and only if the number of *columns* of the first matrix equals the number of *rows* of the second matrix. The number of rows of the product matrix is the same as that of the first matrix, and the number of columns is the same as that of the second matrix. Thinking of each row of the first matrix as a row vector, and each column of the second matrix as a column vector, each cell of the product matrix (if defined) is computed as the inner product of these vectors. Where **C = AB**,

$$c_{ij} = \text{row } i \text{ of } \mathbf{A} \cdot \text{column } j \text{ of } \mathbf{B}$$

For example, suppose that we are given:

$$\mathbf{A} = \begin{bmatrix} 3 & 6 & -4 \\ 8 & -2 & 11 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 5 & 2 & 0 & -3 \\ 8 & 7 & 1 & 12 \\ 13 & -8 & 6 & 5 \end{bmatrix}$$

The product of a 2×3 matrix times a 3×4 matrix is defined, and **C** will be 2×4. Hence eight cells need to be calculated. For example, $c_{1,1}$ is row 1 of **A** $\cdot$ column 1 of **B**, which is $3(5) + 6(8) + (-4)13 = 11$. Cell $c_{1,2}$ is row 1 of **A** $\cdot$ column 2 of **B**, which is $3(2) + 6(7) + (-4)(-8) = 80$. Continuing in this manner we obtain:

$$\mathbf{C} = \begin{bmatrix} 11 & 80 & -18 & 43 \\ 167 & -86 & 64 & 7 \end{bmatrix}$$

While small examples can be done easily by hand, for larger examples matrix multiplication is tedious and easily prone to error. For such examples it is advisable to use a spreadsheet.

The built-in function for matrix multiplication called **MMULT**. Here is how to solve this example using Excel.

1. Enter the data. For example, we could put **A** into the range A1:C2, and **B** into the range E1:H3.

2. Reserve space for the product matrix. For this example, we need 2 rows and 4 columns; we will use range A5:D6. To do this we click on cell A5, and then drag the mouse so that the range A5:D6 (except A5 itself) is shaded medium grey.

3. Enter the formula which calculates the product of the matrix in range A1:C2 with that of the matrix in range E1:H3. In cell A5 we write:

   ```
   =MMULT(A1:C2,E1:H3)
   ```

   The procedure is not as simple as hitting the Enter key. Press the ***Control*** key, and keep it held down, press the ***Shift*** key, and keep it held down, and then press the ***Enter*** key.

After step 3, the solution will appear in the range A5:D6. If the requested matrix multiplication is not defined, the spreadsheet will give an error message.

The Transpose of a Matrix

For every matrix $\mathbf{A}_{m\times n}$ there is a ***transpose matrix*** denoted as $\mathbf{A}^T_{n\times m}$. The numbers in the first row of $\mathbf{A}^T$ come from the numbers in the first column of $\mathbf{A}$. Similarly, the second row of $\mathbf{A}^T$ comes from the second column of $\mathbf{A}$, and so on, with finally row n of $\mathbf{A}^T$ coming from column n of $\mathbf{A}$.

For example:

$$\mathbf{A} = \begin{bmatrix} 2 & 1 & -5 \\ 8 & 9 & 3 \end{bmatrix} \qquad \mathbf{A}^T = \begin{bmatrix} 2 & 8 \\ 1 & 9 \\ -5 & 3 \end{bmatrix}$$

Note that the transpose of the transpose is the original matrix.

$$\left(\mathbf{A}^T\right)^T = \mathbf{A}$$

It is easy enough to do the transpose operation by hand, but it is also available as a spreadsheet function called **TRANSPOSE**.

1. Enter the data. For example, we could put $\mathbf{A}$ into the range A1:C2.

2. Reserve space for the transpose matrix. For this example, we need 3 rows and 2 columns; we will use range E1:F3. To do this we click on cell E1, and then drag the mouse so that the range E1:F3 (except E1 itself) is shaded medium grey.

3. Enter the formula which calculates the transpose of the matrix in range A1:C2. In cell E1 we write:

   `=TRANSPOSE(A1:C2)`

   Press the ***Control*** key, and keep it held down, press the ***Shift*** key, and keep it held down, and then press the ***Enter*** key.

After step 3, the solution will appear in the range E1:F3. The **TRANSPOSE** function is described further in Chapter 5.

The Inverse of a Matrix (Optional)

When a matrix has the same number of rows and columns (i.e $m = n$), it is said to be a square matrix of order n. A special kind of square matrix is the identity matrix, denoted as $\mathbf{I}$. An identity matrix has 1's on the main diagonal, and 0's everywhere else. To illustrate this, the first three identity matrices are:

$$\mathbf{I}_1 = [1] \qquad \mathbf{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

A square matrix $\mathbf{A}$ might have an *inverse* matrix, denoted as $\mathbf{A}^{-1}$, such that:

$$\mathbf{A}\,\mathbf{A}^{-1} = \mathbf{I} \qquad \text{and} \qquad \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$$

A non-square matrix never has an inverse, and not all square matrices have inverses.

There is a formula for finding the inverse (if it exists) of a square matrix of order 2. However, finding the inverse of square matrix of order 3 by hand calculations becomes very tedious. On Excel, the **MINVERSE** function is used to perform matrix inversion. We now use it to invert the following matrix:

$$\mathbf{A} = \begin{bmatrix} 7 & 1 & 5 \\ -2 & 8 & 3 \\ 0.1 & 4 & 6 \end{bmatrix}$$

1. Enter the data. For example, we could put $\mathbf{A}$ into the range A1:C3.

2. Reserve space for the inverse matrix. For this example, we need 3 rows and 3 columns; we will use range E1:G3. To do this we click on cell E1, and then drag the mouse so that the range E1:G3 (except E1 itself) is shaded medium grey.

3. Enter the formula which calculates the inverse of the matrix in range A1:C3. In cell E1 we write:

   ```
   =MINVERSE(A1:C3)
   ```

   Press the ***Control*** key, and keep it held down, press the ***Shift*** key, and keep it held down, and then press the ***Enter*** key.

After step 3, the solution will appear in the range E1:G3. If no inverse matrix exists, then `NUM!` will appear in each cell of the matrix. The inverse matrix is:

$$\mathbf{A}^{-1} = \begin{bmatrix} 0.163414 & 0.06355 & -0.16795 \\ 0.055833 & 0.188379 & -0.14072 \\ -0.03995 & -0.12665 & 0.263277 \end{bmatrix}$$

Solving Linear Equations (Optional)

Here is how to solve a system of linear equations using Excel:

1. Put the equations into the form $\mathbf{Ax} = \mathbf{b}$.
2. Use the MINVERSE function to try to find $\mathbf{A}^{-1}$. There is a unique solution for $\mathbf{x}$ if and only if the inverse matrix exists.
3. If the inverse exists use the MMULT function to compute $\mathbf{x} = \mathbf{A}^{-1}\,\mathbf{b}$.

We wish to solve the following linear system.

$$\begin{aligned} 3X_1 + 2X_2 + 7X_3 &= 53 \\ 5X_1 - 4X_2 + 8X_3 &= 26 \\ 6X_1 + 10X_3 &= 62 \end{aligned}$$

Putting it into matrix form we obtain:

$$\begin{bmatrix} 3 & 2 & 7 \\ 5 & -4 & 8 \\ 6 & 0 & 10 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 53 \\ 26 \\ 62 \end{bmatrix}$$

If the inverse exists, we need to find

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 5 & -4 & 8 \\ 6 & 0 & 10 \end{bmatrix}^{-1} \begin{bmatrix} 53 \\ 26 \\ 62 \end{bmatrix}$$

Using the Excel MINVERSE function to perform the matrix inversion we obtain:

$$\begin{bmatrix} 3 & 2 & 7 \\ 5 & -4 & 8 \\ 6 & 0 & 10 \end{bmatrix}^{-1} = \begin{bmatrix} -0.909091 & -0.454545 & 1.00 \\ -0.045455 & -0.272727 & 0.25 \\ 0.545455 & 0.272727 & -0.50 \end{bmatrix}$$

Therefore we wish to solve:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} -0.909091 & -0.454545 & 1.00 \\ -0.045455 & -0.272727 & 0.25 \\ 0.545455 & 0.272727 & -0.50 \end{bmatrix} \begin{bmatrix} 53 \\ 26 \\ 62 \end{bmatrix}$$

Using the Excel MMULT function to multiply the two matrices we obtain:

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$$

The unique solution is $X_1 = 2$, $X_2 = 6$, and $X_3 = 5$.

1.3 Example – Mobile Telephone Plans

We use this example as a way to illustrate the paradigm of decision modeling, and to illustrate the use of spreadsheets.

1.3.1 Problem Identification

Alison has decided to buy a mobile telephone, partly for safety in case of a breakdown in an isolated area, but also because of the convenience that it will provide. She's not concerned about the initial cost of the telephone itself, especially when some mobile telephone companies give the phones away in order to attract business. However, she is concerned about the monthly operating cost, especially since she would use the phone almost entirely during working hours Monday to Friday. Some of her calls will be to long-distance (but in-the-country) destinations. In addition to voice calls, she also wants to be able to send and receive text messages. She is not interested in using a mobile phone to connect to the Internet.

Going to a mobile phone company store, she finds a brochure that gives details about eight plans. She easily narrows it down to two plans, because the cheapest of the eight plans does not include text messaging, and the five most expensive plans include data (i.e. connecting to the Internet) that she doesn't wish to pay for.

These two plans both offer unlimited text, picture, and video messages. Also, they both offer unlimited local calls in the evenings and on weekends. The plans differ in price, and in the number of Monday to Friday daytime local calls that are included in the price. One plan costs $35 per month and includes 200 minutes per month of local calls, while the other costs $42 per month but includes 1000 local minutes. For either plan, the indicated number of local minutes can be made into anywhere (local or long-distance) minutes for an extra $10 per month. For either plan, extra minutes (local or long-distance) cost $0.50 per minute.

Because of the all-or-nothing nature of the base costs, there are effectively four plans of interest:

1. Plan 1 has 200 local minutes and costs $35 per month.
2. Plan 2 has 1000 local minutes and costs $42 per month.
3. Plan 3 has 200 anywhere minutes and costs $35 + $10 = $45 per month.
4. Plan 4 has 1000 anywhere minutes and costs $42 + $10 = $52 per month.

In addition, for all four plans, extra minutes (local or long-distance) cost 50 cents per minute.

Suppose that Alison wishes to make 140 minutes of daytime weekday local calls, and 80 minutes of long-distance calls. We will build a spreadsheet to figure out the best plan for any amount of local and long-distance minutes, but with just 140 and 80 in mind, we can easily work out the cost of each plan by hand.

Using Plan 1, which has a base cost $35, all her local calls are "free", but she pays an extra 80 @ $0.50 = $40 in long-distance charges, for a total of $75. Plan 2 only makes her worse off; she'll pay $42 + $40 = $82. Plan 3 with a base cost of $45 covers her for 200 of the 220 minutes, so she pays an extra 20 @ $0.50 = $10 for a total of $55. Finally for $52 Plan 4 covers all her calls. Clearly, based on the stated intended usage, Plan 4, which gives 1000 anywhere minutes for $52 per month, is the cheapest plan.

Now instead of speaking of specific numbers like 140 local minutes and 80 long-distance minutes, let's suppose that she expects to make a minutes of local calls, and b minutes of long-distance calls. Here, a and b are not variables, but rather they are parameters, that is, they are fixed for a given example, but can change from one example to another. Now let's work out the total cost for each plan as a function of a and b.

Plan 1 If $a \leq 200$, the cost will be $\$35 + \$0.50b$. However, if $a > 200$, then she must pay an additional $\$0.50(a-200)$. We can put these expressions together into one, by using a finding the maximum of $a-200$ and 0. The total cost is:

$$35 + 0.5\max\{a-200, 0\} + 0.5b$$

Equivalently, we can write:

$$35 + 0.5\max\{a+b-200, b\}$$

Plan 2 It's easier to find the cost if we simply look at what is different from Plan 1. The base cost is \$42, and the plan limit for local calls is 1000 minutes, hence the cost is:

$$42 + 0.5\max\{a+b-1000, b\}$$

Plan 3 In this plan as long as $a+b$ doesn't exceed 200 there is no charge beyond the basic \$45; there is a 50 cent per minute charge for minutes over this limit. The total cost is:

$$45 + 0.5\max\{a+b-200, 0\}$$

Plan 4 This is similar to Plan 3, but with a base charge of \$52, and 1000 anywhere minutes. The total cost as a function of a and b is:

$$52 + 0.5\max\{a+b-1000, 0\}$$

The objective is of course cost minimization. For any particular a and b, we wish to:

$$\begin{aligned}&\min\{35 + 0.5\max\{a+b-200, b\}; 42 + 0.5\max\{a+b-1000, b\};\\ &45 + 0.5\max\{a+b-200, 0\}; 52 + 0.5\max\{a+b-1000, 0\}\}\end{aligned}$$

1.3.2 Model Solution

What we have done so far is make an *algebraic* model of the problem. Because this problem with specific numbers $a = 140$ and $b = 80$ is simple, we solved it by hand. However, with the costs of the plans now in terms of a local minutes and b long-distance minutes, it is useful to make a *spreadsheet* model to calculate the cost of each plan for several values of a and b.

In the model below we start with the values that we considered earlier, in which $a = 140$ and $b = 80$. The numbers in cell range F6:F9 and cell H8 are calculated by Excel; the other numbers are input data. Also, Excel determines and then shows the best plan in cell H9. Note that the input data has not been embedded in the spreadsheet formulas. By doing it as shown, if a change needs to be made, for example, suppose that the extra-minute charge increases to 60 cents per minute, then all we need to do is change one cell (cell H3 in the example) from \$0.50 to \$0.60. Also, doing it this way means that the cost per minute is transparent to anyone seeing the spreadsheet.

	A	B	C	D	E	F	G	H
1		**Choosing a Mobile Telephone Plan**						
2								
3		Local (a)	140	Long-distance (b)		80		\$0.50
4								
5	Plan	Base Cost		Minutes		Total Cost		
6	1	\$35.00	local	200		\$75.00		**Best**
7	2	\$42.00	local	1000		\$82.00		**Plan**
8	3	\$45.00	anywhere	200		\$55.00		\$52.00
9	4	\$52.00	anywhere	1000		\$52.00		Plan 4

The following graphic shows one way of writing the required formula for each plan. The formula was written out in full for cell F6 as shown. Because the formula has a slightly different form for Plans 3 and 4, we could not copy F6 into F6:F9. Instead, F6 was copied into cells F6:F8. Then, the formula is cell F8 was modified at the end, replacing `$F$3` with 0. The modified cell F8 was then copied into F8:F9.

	F
5	Total Cost
6	=B6+\$H\$3*MAX(\$C\$3+\$F\$3-D6,\$F\$3)
7	=B7+\$H\$3*MAX(\$C\$3+\$F\$3-D7,\$F\$3)
8	=B8+\$H\$3*MAX(\$C\$3+\$F\$3-D8,0)
9	=B9+\$H\$3*MAX(\$C\$3+\$F\$3-D9,0)

To have Excel show the best plan, we have used a nested IF statement to figure out which plan is associated with the least cost. By putting the result of the IF statement in quotation marks, Excel will show verbatim what is inside the marks.

	H
6	**Best**
7	**Plan**
8	=MIN(F6:F9)
9	=IF(F6<=H8,"Plan 1",IF(F7<=H8,"Plan 2",IF(F8<=H8,"Plan 3","Plan 4")))

The nested IF is compatible with all versions of Excel. In 2016, a new **IFS** function was introduced. Using this function we can accomplish what the nested IF does. The syntax for this example is:

	H
9	=IFS(F6<=H8,"Plan 1",F7<=H8,"Plan 2",F8<=H8,"Plan 3",F9<=H8,"Plan 4")

The developer of the model, whether using a calculator or a spreadsheet, must make the recommendation clear. The customer of the model (in this case, Alison) might not be familiar with spreadsheets, so the emphasis should be on giving the recommendation:

Recommendation

Based on expecting to need 140 weekday daytime local minutes, and 80 long-distance minutes, Alison should sign up for Plan 4 ($52 for 1000 anywhere minutes), at a cost of $52 per month.

Changing a and b

Having set up the spreadsheet, it will easily calculate whatever numbers we give it. Suppose that we change a to 30 minutes and b to 70 minutes. In an instant the spreadsheet updates and we see that the least-cost plan now is Plan 3 at a cost of $45.00.

	A	B	C	D	E	F	G	H
1		**Choosing a Mobile Telephone Plan**						
2								
3		Local (a)	30	Long-distance (b)		70		$0.50
4								
5	Plan	Base Cost		Minutes		Total Cost		
6	1	$35.00	local	200		$70.00		**Best**
7	2	$42.00	local	1000		$77.00		**Plan**
8	3	$45.00	anywhere	200		$45.00		$45.00
9	4	$52.00	anywhere	1000		$52.00		Plan 3

By playing around with the two input parameters, we can see the minimal cost solution for any pair (a, b). This is a primitive form of *sensitivity analysis*, in which a parameter of the model is varied to examine the effect (if any) on the recommended solution.

Implementation

Models only approximate reality. Sometimes, things can be left out because they don't affect the choice. For example, we have ignored taxes. Whatever the tax rate, the cheapest alternative is still the cheapest after taxes have been included. On the other hand, a model cannot capture every nuance, even if it might change the optimal choice. It may be that some plans have extra features like call forwarding, but some plans do not. If we try to capture this in the model, it will quickly become very big. For this reason, the recommended solution is only optimal for the model, and is not necessarily best at solving the original problem. To complete the paradigm, we should go back to Alison to see if she is happy with the recommended plan.

Commentary

The four phases of the management science paradigm are not totally distinct. When we had completed the algebraic model, we saw that it was useful to build another model, this one using a spreadsheet, so that we could solve it.

This model only involved cost, so we found the alternative with the minimum cost. In many management science examples, however, we seek the alternative with the highest profit.

1.4 Break-Even Analysis

Introduction

Break-even analysis compares two alternatives to tell us when we should switch from one to the other.

Example 1 A company is not currently making any umbrellas, but believes that with a capital expenditure of \$12,000 they can pay for the required investment in machinery and training. With this investment made, they could produce umbrellas at a marginal cost of \$8.00 each which they would sell to a wholesaler for \$10.00 each.

If they make and sell x umbrellas, their profit will be $10x - 8x - 12,000$, i.e. $2x - 12,000$. Not making the investment has a profit of 0. They are better off making the investment if $2x - 12,000 > 0$, better off doing nothing if $2x - 12,000 < 0$, and they are indifferent if $2x - 12,000 = 0$. This latter case gives the value of x which is the break-even point (BEP). Solving we obtain BEP = 6,000 umbrellas.

Example 2 A business takes all its photocopying needs to a nearby copy service, which charges 10 cents per page. They are considering renting their own machine for \$420 per month, which would operate with a variable cost of only 4 cents per page.

Based on a volume of x copies per month Alternative 1 (continue to use the copy service) would cost $0.1x$, while Alternative 2 (rent their own photocopying machine) would cost $420 + 0.04x$. Break-even analysis sets the two costs equal to each other to determine the break-even quantity:

$$\begin{aligned} 0.1x &= 420 + 0.04x \\ 0.06x &= 420 \\ x &= 7000 \end{aligned}$$

Hence at the break-even point of BEP = 7000 copies per month, the company would be indifferent between the two alternatives. For $x < 7000$, they should continue to go to the copy service, and if $x > 7000$, they should rent their own photocopy machine.

Mobile Phone Plans

We can apply break-even analysis to the mobile telephone plan problem, but there are two complications. The first is that some of the expressions use a MAX function. The second is that there are four plans, but break-even analysis compares one alternative with another one. To compare each plan with each other plan would require six comparisons (1 and 2, 1 and 3, 1 and 4, 2 and 3, 2 and 4, and 3 and 4). We shall just do the first two of these six comparisons.

Plan 1 vs. Plan 2 Plans 1 and 2 have the same cost when:

$$35+0.5\max\{a+b-200,b\}=42+0.5\max\{a+b-1000,b\}$$

While we could just find the break-even point, a more useful analysis finds when one plan is better than the other. Plan 1 is better than (i.e. cheaper than) Plan 2 when:

$$35+0.5\max\{a+b-200,b\}\leq 42+0.5\max\{a+b-1000,b\}$$

Subtracting 35 from each side gives us:

$$0.5\max\{a+b-200,b\}\leq 7+0.5\max\{a+b-1000,b\}$$

Multiplying both sides by 2 we obtain:

$$\max\{a+b-200,b\}\leq 14+\max\{a+b-1000,b\}$$

We can extract b from both parts of the max expression, to give :

$$\max\{a-200,0\}+b\leq 14+\max\{a-1000,0\}+b$$

Now we subtract b from both sides to obtain:

$$\max\{a-200,0\}\leq 14+\max\{a-1000,0\}$$

Case 1 Suppose that $a\leq 200$. Hence $a-200\leq 0$, and therefore $\max\{a-200,0\}=0$. Also, $\max\{a-1000,0\}=0$. Therefore $\max\{a-200,0\}\leq 14+\max\{a-1000,0\}$ reduces to $0\leq 14+0$, which is always true. This means that whenever $a\leq 200$, Plan 1 is better than Plan 2.

Case 2 Now suppose that $a \geq 1000$. The condition now requires that:

$$a - 200 \leq 14 + a - 1000$$

This reduces to $800 \leq 14$, which is a contradiction. This means that, if $a \geq 1000$, Plan 1 is never better than Plan 2.

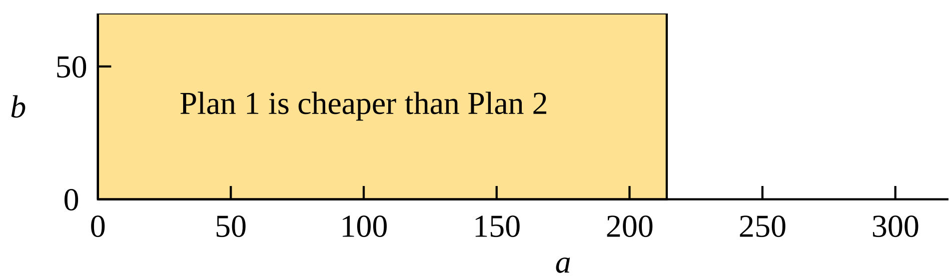
Case 3 The only other possibility is $200 \leq a \leq 1000$. With this assumption, $\max\{a - 200, 0\} \leq 14 + \max\{a - 1000, 0\}$ simplifies to:

$$a - 200 \leq 14 + 0$$

i.e. $a \leq 214$.

Overall Hence, Plan 1 is preferred over Plan 2 if $a \leq 214$, Plan 2 is preferred over Plan 1 if $a \geq 214$. and when $a = 214$ Plans 1 and 2 cost the same. Note that b affects the cost of both plans, but does so equally, and hence b does not help determine the switchover point.

Part of the region of infinite size where Plan 1 is better than Plan 2 can be shown graphically, putting a on the horizontal axis, and b on the vertical axis.



Plan 1 vs. Plan 3 Plan 1 is better (i.e. cheaper) than Plan 3 when:

$$35 + 0.5 \max\{a + b - 200, b\} \leq 45 + 0.5 \max\{a + b - 200, 0\}$$

First, we subtract 35 from both sides of the inequality:

$$0.5 \max\{a + b - 200, b\} \leq 10 + 0.5 \max\{a + b - 200, 0\}$$

Now we multiply both sides by 2:

$$\max\{a + b - 200, b\} \leq 20 + \max\{a + b - 200, 0\}$$

Case 1 Suppose that $a+b-200 \leq 0$, i.e. $a+b \leq 200$. The expression simplifies to $b \leq 20$.

Case 2 Now suppose that $a+b-200 \geq b$, i.e. $a \geq 200$. The expression simplifies to:

$$a+b-200 \leq 20+a+b-200$$

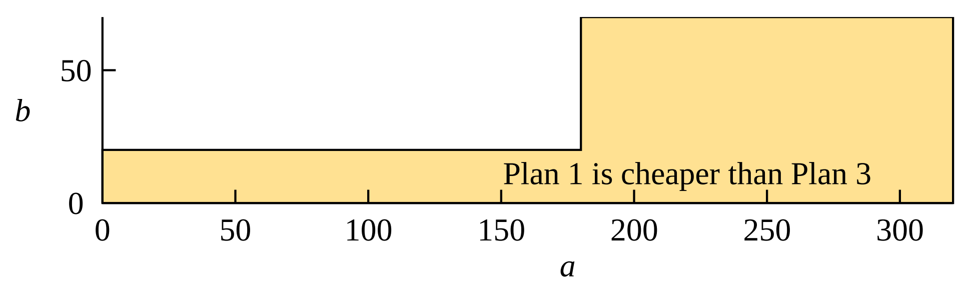
which further simplifies to $0 \leq 20$, which is always true. In other words, when $a \geq 200$, Plan 1 is always better than Plan 3.

Case 3 The only other possibility is $0 \leq a+b-200 \leq b$, i.e. $a \leq 200$. With this assumption we obtain:

$$b \leq 20+a+b-200$$

which simplifies to $a \geq 180$.

Overall Taking these cases into consideration, Plan 1 is better than Plan 3 whenever $a \geq 180$, or $b \leq 20$. Another way of saying this is that Plan 3 is better than Plan 1 provided that both $a \leq 180$ and $b \geq 20$. The graph of the region where Plan 1 is better than Plan 3 is:



1.5 Why Decision Modeling is Important

1.5.1 Using Resources Efficiently

Decision Modeling is part of a wider subject called *Operational Research* in Canada, or *Operations Research* in the U.S.A. (O.R. is the initialism for both).

Because of its heritage from many sources,[4] O.R. is also called *Management Science*, or *Decision Analysis*. There's a new word being bandied about, and that is *Analytics*. As catchy as that word is, it operates more at the boundary of O.R. and Statistics, so this course adopted the Decision Modeling name. This course deemphasizes problem solution, leaving difficult problems to the computer, so we didn't call it operational research. However, a student can take electives to become proficient in O.R.

O.R. is about applying mathematical techniques to use resources more efficiently. Suppose that a truck has to leave a warehouse, go to customers 1, 2, and 3, and then return to the warehouse. Suppose that the distance in kilometres from the warehouse to 1, 2, and 3, is 3.5, 4.5, and 5.5 respectively, 1 to 2 and 3 is 4.0 and 4.4 respectively, and 2 to 3 is 2.7, and all distances are symmetric (e.g. 1 to the warehouse is 3.5 km). Here are these distances written in tabular form:

	W	1	2	3
W	—	3.5	4.5	5.5
1	3.5	—	4.0	4.4
2	4.5	4.0	—	2.7
3	5.5	4.4	2.7	—

There are only six ($= 3 \times 2 \times 1$) possible routes.[5] One way to route the truck would be to simply go from the warehouse to 1, then to 2, then to 3, and then back to the warehouse, for a total distance of $3.5+4.0+2.7+5.5 = 15.7$ km. However, we can find a better way (i.e. lower total distance travelled) by inspection, or if need be, by a complete enumeration of all six routes. Doing this we can see that the best way to route the truck is to go from the warehouse to 1, then to 3, then to 2, then back to the warehouse, for a total distance of $3.5+4.4+2.7+4.5 = 15.1$ km. The best solution of 15.1 km is 0.6 km lower (or about 3.8% lower) than the 15.7 km solution. The truck saves on fuel and wear-and-tear. Also, it might make the truck driver more productive because the truck will return to the warehouse a bit sooner. The company is staying competitive by using its resources wisely.

In a real-life situation such an example might be complicated by issues such as one-way streets, or difficult left-turns, but these things just make the distance table non-symmetric. However, one thing that does make things more difficult is to

[4] For an overview of how OR came about, see Saul I. Gass, 2011, Model World: On the Evolution of Operations Research, *Interfaces*, **41**, No. 4, pp. 389-393.

[5] Because of the symmetry of the distance table, only three (half of six) routes need to be evaluated.

increase the number of customers. This example is a type of *travelling salesman problem*, which has been well-studied.[6] In general, where there is a warehouse with deliveries to be made to n customers, there are n factorial (written as $n!$) ways to route the truck.[7] Complete enumeration of $n!$ routes is out of the question when n is large. Instead, we need to make a mathematical model and then solve it with an intelligent algorithm; even so, this can be still be difficult to do.

Even for the vast majority of readers of this document who will never become O.R. professionals, there is a great deal to be gained from studying decision modeling, just as accounting majors are helped by studying marketing, and marketing majors are helped by knowing something about accounting. Firstly, it is of great benefit just to be aware that something can be improved, for otherwise it never will be. Secondly, we see that with simple models we can obtain more profitable solutions at very low incremental cost, because most of them can be done on a spreadsheet. Thirdly, a student who is aware of what could be possible in terms of optimization can interact with specialist professionals trained in O.R. be they in-house technical people, or outside consultants.

For those who plan to major in O.R. joining a professional society as a student member would be a good place to begin.

1.5.2 Professional Societies

The professional society for O.R. in Canada is the Canadian Operational Research Society, or CORS for short. Information about CORS may be obtained from http://www.cors.ca. In the United States, the Institute for Operations Research and the Management Sciences, abbreviated as INFORMS, is the world's largest O.R. society. Their website is at https://www.informs.org/. Both CORS and INFORMS are part of IFORS (http://ifors.org/), the International Federation of Operational Research Societies.

CORS and INFORMS co-operate by holding a joint conference about once per decade. In other years, CORS holds a conference on its own or with another organization, at various locations across Canada.

The Administrative Sciences Association of Canada (http://www.asac.ca) is a professional society serving all fields of business education. It organizes an annual conference with sessions organized for all these fields, which includes Management Science.

[6] See, for example, https://en.wikipedia.org/wiki/Travelling_salesman_problem.

[7] With symmetry, the number to evaluate is cut in half.

1.5.3 OR Applications and Awards

Students often wonder where the material of this course would be used in real life. Both the CORS and INFORMS websites give examples of such applications. Also, the *INFORMS Journal on Applied Analytics*[8] is a good source of applications. This journal is the most applied O.R. journal, and the easiest to read, though in some places it uses mathematics that will be advanced for someone who is just beginning to study decision modeling.

CORS and INFORMS offer prizes for excellence in O.R. Information about the CORS prizes is available from http://www.cors.ca/?q=content/practice-prize-competition. Some of the areas of research associated with the awards include: insurance fraud; transportation; health care; scheduling of sports; designing electoral districts; and production planning. Information about the INFORMS prizes for excellence is available at https://www.informs.org/Recognizing-Excellence.

1.6 Problems for Student Completion

1.6.1 Spreadsheet Formula Exercises

For each of the following, solve in Excel, making one file, with a tab for each part. A tab named "Sheet 1" will appear at the bottom-left. If part (a) has been solved on this sheet, the tab can be renamed to something like "1.6.1 (a)". By clicking on the circled plus sign to the right, a new sheet will open, on which part (b) can be solved.

(a) An income-tax credit for charitable donations is calculated at the lowest tax rate on the first \$200, and at the highest rate on anything exceeding that amount. The lowest tax rate of 21.5% is in cell A2, and the highest tax rate of 48.5% is in cell A5. An individual's total charitable donations of \$3700 are in cell C2. Cells A1, A4, and C1 are used as labels for these three things, and the \$200 is entered into cell B2. With the label "Tax Credit" in cell C4, put the appropriate formula in cell C5 to calculate this person's tax credit.

(b) A course has two midterm tests with the weight of 20% each being in cell G2, and the weight of the final exam in cell G3 is =1−2*G2. However, the professor will drop the lower test mark and add the weight to the final exam if and only if this would help the student. The names of the students are in

[8] Previously named *Interfaces*.

column A, starting in row 3, with the label "Student Names" in row 1 of this column. The marks for test 1, test 2, and the final exam are in columns B, C, and D respectively, with the corresponding labels in row 1 and the numbers beginning in row 3. Find a formula in column E which calculates the final mark for every student and rounds it to the nearest integer. Solve this in Excel using the following data:

Student Names	Test 1	Test 2	Final Exam
Bartlett, Joanna	37	65	73
Chan, Mia	82	86	81
Duval, Pierre	72	56	64

(c) A mining company uses a grid system in which a location at a given depth below the surface is identified as (a,b), being a metres east of a reference point and b metres north of it. At a depth of 500 metres below the ground, they wish to construct a tunnel from point $(103,296)$ to $(345,237)$. Use Excel to calculate the length of the tunnel.

1.6.2 Phone Plans

Consider the mobile telephone plans named Plan 1 and Plan 4, which are described beginning on page 19.

(a) Determine analytically the values of a and b for which Plan 1 is better (i.e. cheaper) than Plan 4.

(b) With a on the horizontal axis, and b on the vertical axis, show the region found in part (a).

1.6.3 Cargo Plane Loading Problem

Two types of big boxes are about to be loaded onto a small cargo plane. A Type 1 box has a volume of 2.9 cubic metres (m^3), and a mass of 470 kilograms (kg), while a Type 2 box has a volume of 1.8 m^3 and a mass of 530 kg. There are six Type 1 boxes and eight Type 2 boxes waiting to be loaded. There is only one cargo plane, and it has a volume capacity of 15 m^3 and a mass capacity of 3600 kg. Obviously, not all the boxes can be put onto the plane, therefore suppose that the objective is to maximize the value of the load. We will consider the following three situations: (i) both type of boxes are worth \$400 each; (ii) a Type 1 box is worth \$600, and a Type 2 box is worth \$250; and (iii) a Type 1 box is worth \$300, and a Type 2 box is worth \$750.

Later in this course we shall see an efficient approach for solving this type of problem, but for now we use the following simple approach:

(a) Let X and Y represent the number of Type 1 and Type 2 boxes respectively which are put onto the plane. Where X and Y are of course positive integers (including 0), determine all the feasible combinations (X,Y), using a spreadsheet to help with the calculations. (To be feasible the total volume carried must be $\leq 15m^3$, and the total mass carried must be ≤ 3600 kg.)

(b) Consider a combination found in (a) which can be augmented by adding one box (of either type) with the capacities still not being exceeded. One example is $(3,1)$, i.e. three Type 1 boxes, plus one Type 2 box, because $(3+1,1) = (4,1)$, which is feasible, would be a better solution, as would the feasible solution $(3,1+1) = (3,2)$. This combination $(3,1)$ (and all others like it) is therefore trivially sub-optimal, because we would obtain more money by adding the extra box.

Therefore, we should narrow the search by looking only at the feasible combinations which are so near the limit of either the mass or volume capacity that putting one more box (of either type) onto the plane would make it unable to fly. Mathematically, these are the combinations for which (X,Y) is feasible, but neither $(X+1,Y)$ nor $(X,Y+1)$ is feasible. Find these combinations.

(c) Make a spreadsheet in which the alternatives are the combinations from (b), and which has two cells reserved for the value of each type of box. Use the spreadsheet to determine, for each of the three financial scenarios, how many boxes of each type are carried, and the value of the load.

Work on this on your own and come up with your own method. If you're stuck after 15 minutes or so, then look at the hints which follow.

Hints

Obviously carrying no boxes is feasible, so this is a good starting point. This solution is represented as $(X,Y) = (0,0)$. We could then determine using trial-and-error if (0,1) is feasible, and if so, then see if (0,2) is feasible, and so on. A faster way, however, is to start by fixing $X = 0$, and then find the largest value for Y. There are three restrictions: we cannot exceed the volume available; we cannot exceed the mass available; and Y must be an integer. When $X = 0$ the volume available is of course the full 15 m^3, and the mass available is 3600 kg. Each unit of Y (each Type 2 box) takes up 1.8 m^3 and 530 kg, therefore Y is the largest integer such that both $1.8Y \leq 15$ and $530Y \leq 3600$. Hence $Y \leq 8.333...$, and $Y \leq 6.792....$ Hence the most that Y can be is 6. Therefore all combinations (X,Y) = (0,0), (0,1), (0,2), (0,3), (0,4), (0,5), and (0,6) are feasible.

Now suppose that $X = 1$. This takes up 2.9 m^3 and 470 kg, therefore the type 2 boxes can use up to $15 - 2.9 = 12.1m^3$ and up to $3600 - 470 = 3130$ kg. Based on this, it can be seen that Y can be at most 5. Keep repeating this for higher values of X until no more type 1 boxes can be carried, even if no type 2 boxes are carried. You should find a total of 26 feasible combinations. Using the rules of part (b), we see that the search can be limited to just five combinations.

A spreadsheet to do the calculations for part (a) could begin as shown on the next page.

Find the formula for each of the cells B4, C4, D4, E4, and F4. Do not hard-encode the data into each cell, but rather use absolute cell addresses. For F4, the INT function is needed. The range B4:F4 is then copied to the rows below. There will be a row in which it and all subsequent rows contain one or more negative numbers; this means that the corresponding value of X is infeasible.

	A	B	C	D	E	F	G	H	I	J
1	Cargo Plane Loading Problem			X = number of Type 1 boxes			Y = number of Type 2 boxes			
2										
3		Type 1	Type 1	Type 2	Type 2		Plane Volume Capacity (m^3)			15
4		Volume	Mass	Volume	Mass		Plane Mass Capacity (kg)			3600
5		2.9	470	1.8	530					
6										
7	X	Remaining	Remaining	Max. Y	Max. Y	Overall				
8		Volume	Mass	(Volume)	(Mass)	Max. Y				
9										
10	0									
11	1									
12	2									
13	3									
14	4									
15	5									
16	6									

Chapter 2

Elementary Modeling

We begin with an example involving cement production to illustrate the topic of *linear optimization* in the context of the maximization of an objective. This model is then solved graphically. We then consider variations which lead to a more general understanding of what linear optimization is, and then consider an extension to the cement example, and provide its graphical solution. Next, a diet problem illustrates linear optimization when the objective is minimization. This too is solved graphically. Finally, we show how to solve these problems using the Solver on Excel.

2.1 Example – Cement Problem

2.1.1 Problem Description

A cement company makes two types of cement, which they market under registered tradenames, but for our purposes we will simply call them Type 1 and Type 2. Cement is sold by the Tonne (a Tonne is 1000 kilograms), and production is measured in Tonnes per Day, abbreviated as TPD. The company has contractual sales obligations to produce at least 40 TPD of Type 1 cement, and at least 30 TPD of Type 2 cement.

The physical capacity of the plant, which is governed by such things as conveyor belt speed, storage size, and so on, is limited to 200 TPD. A new labour agreement has increased the length of breaks, and restricts and makes more costly the use of overtime. The company therefore wishes to find its best production plan using the new work rules with everyone working a 40 hour week. Work is measured in this company by the labour-hour, which is one person working for one hour. Each type of cement is made in three departments, labeled A, B, and C. To make each Tonne of Type 1 cement requires three labour-hours in Department A, one and a half labour-hours in Department B, and four labour-hours in Department C. The amounts of work per Tonne of Type 2 cement are two, five, and six labour-hours in Departments A, B, and C respectively.

Based on the current authorized strength in each department, and factoring in allowances for breaks, absenteeism, and so on, Department A has 585 labour-hours available each day. Departments B and C are allowed to use up to 500 and 900 labour-hours per day respectively. These are the most they can use for the making of cement. If a department has some time leftover (i.e. if the time to make the cement is less than the number of labour-hours available), then the workers will be idle for a few minutes at the end of the day. The three departments require workers with very different training and skills, so the possibility of transferring employees from one department to another is not something that is factored into the planning process.

Taking the market price of each type of cement and from this subtracting all the variable costs of making the cement leaves the company with a profit of $8 per Tonne of Type 1 cement, and $10 per Tonne of Type 2 cement. There are also fixed costs (taxes, security, and so on) which total $1400 per day. The company wants to know how much should be produced of each type of cement, so that the profit is maximized.

2.1.2 Making a Model

Verbal, Algebraic, and Spreadsheet Models

Someone has already gone into the cement plant to obtain the relevant facts and from this research a verbal model has been made, which appears as the "Problem Description". This model is complete in that the final sentence states the essence of the problem, and gives the objective. Often, only the data is provided with a general question of the "what should the company do?" variety.

In order to solve the problem, we need to transform the verbal model into an *algebraic model*. Models with just two variables can be solved graphically, but of course this is of limited practical use. Algebraic models can be solved by a software package designed for this purpose, up to a size limit set by the writers of the software. Another option is to transform the algebraic model into a spreadsheet modelon Excel. It can then be solved by using the Solver, as we shall later see. Indeed, for a very simple problem like the cement problem, one can bypass the algebraic model and go directly to the spreadsheet model. However, this shortcut will not help us for more complex models, so we will not take this route.

Definition of the Variables

In beginning to make an algebraic model, we wish to determine the unknowns which will be represented using variables. The emphasis here is to focus in on the unknowns which are at the heart of the problem, and to skip those things which can easily be determined once the essential unknowns have been determined. In this problem, these unknowns come from the last sentence of the problem description: the number of TPD of Type 1 cement that should be made; and the number of TPD of Type 2 cement that should be made. Everything else, such as the total profit, or the idle time (if any) in one of the departments, can be determined if we know these two essential things. With just two unknowns we could label them X and Y, but it is more common to use subscripts, calling them X_1 and X_2.[1] This way of labelling the unknowns is what is required when we consider realistically sized models, which can have thousands of variables. Hence we have:

$$
\begin{aligned}
X_1 = &\quad \text{the number of TPD of Type 1 cement made} \\
X_2 = &\quad \text{the number of TPD of Type 2 cement made}
\end{aligned}
$$

It is very important that the definitions of the variables be made as clearly as possible. For example, a shorthand such as "$X_1 =$ Type 1" is *not* acceptable.

The Objective Function

We now need to write an expression for the profit in terms of the variables. Looking at the Type 1 cement alone, one Tonne gives a contribution of \$8 to the profit. Since we are producing X_1 TPD, the daily profit from the production of Type 1 cement is $8X_1$. Similarly, the daily profit from the production of Type 2 cement is $10X_2$. Putting these together we have $8X_1 + 10X_2$. The \$1400 in daily fixed costs needs to be subtracted from this expression, but most traditional software (before spreadsheets) is not set up to handle this. Therefore we omit subtracting it for now, but we can easily subtract it at the very end when everything else has been calculated. We write the word *maximize* in front of the expression, because that is

[1]The convention in this document is to use capital names for variables, and small letters for parameters. Hence c_1X_1 refers to the product of parameter c_1 and variable X_1.

the objective in this situation. The word *maximize* is often abbreviated to simply *max*. What we call the *objective function* is:

$$\text{maximize } f(X_1, X_2) = 8X_1 + 10X_2$$

In this document we refer to the value of the objective function as OFV (for objective function value). (A more traditional (but less intuitive) symbol is Z.) It is conventional to omit the "$f(X_1, X_2) =$". Hence we simplify the objective function to:

$$\text{maximize } \; 8X_1 + 10X_2$$

The Constraints

The objective function is subject to a set of constraints which represent, in this example, the minimum sales contract requirements, the limit on total production, and the limit on labour availability in each of the three departments. Also present in this and in almost every linear optimization model are non-negativity restrictions on the variables.

Non-Negativity Restrictions Since we cannot produce a negative quantity of cement, we require that X_1 be greater than or equal to 0, and that X_2 be greater than or equal to 0. When writing the algebraic model, we will indicate this by writing $X_1 \geq 0$ and $X_2 \geq 0$ at the end, or in short form simply $X_1, X_2 \geq 0$. (Most software programs assume these restrictions and therefore they do not need to be explicitly entered.) By convention, this short form is only used for the non-negativity restrictions; it is not used for the other constraints.

Three Easy Constraints The first three constraints are quite easy. Their sales contracts for 40 TPD of Type 1 cement and 30 TPD of Type 2 cement means that we must have $X_1 \geq 40$ and $X_2 \geq 30$. Theoretically, these constraints make the non-negativity restrictions superfluous, but we keep them anyway. This is because the model might later change – should the sales constraints be removed, then the non-negativity restrictions would become the new lower bounds on the variables. The third constraint that the total production cannot exceed 200 TPD is represented by $X_1 + X_2 \leq 200$. So far the constraint list is:

$$\begin{array}{rcccccc}
\text{Type 1 Sales} & X_1 & & & \geq & 40 \\
\text{Type 2 Sales} & & & X_2 & \geq & 30 \\
\text{Total Production} & X_1 & + & X_2 & \leq & 200
\end{array}$$

The Labour Constraints Now we determine the three labour constraints, one for each department. The data for these constraints is written both from a product perspective and a departmental perspective. From the product perspective we have:

> To make each Tonne of Type 1 cement requires three labour-hours in Department A, one and a half labour-hours in Department B, and four labour-hours in Department C. The amounts of work per Tonne of Type 2 cement are two, five, and six labour-hours in Departments A, B, and C respectively.

From the departmental perspective we have:

> Based on the current authorized strength in each department, and factoring in allowances for breaks, absenteeism, and so on, Department A has 585 labour-hours available each day. Departments B and C could use up to 500 and 900 labour-hours per day respectively.

It may be helpful to put all this data into a table with two rows, one for each type of cement, and three columns for the labour-hours to make one Tonne of Type 1, the labour-hours to make one Tonne of Type 2, and the number of labour-hours available per day. Note that in this example, the data from the problem description go into the *columns*. (Be careful about this, in other problems some of the data might go into the rows).

	Labour-Hours per Tonne		Labour-Hours
Department	of Type 1 Cement	of Type 2 Cement	Available each day
A	3	2	585
B	1.5	5	500
C	4	6	900

In each department, the labour-hours (LH) used cannot exceed the labour-hours available. Let's look at Department A in particular.

$$\begin{aligned} \text{LH used} &\leq \text{LH available} \\ \text{LH to make Type 1} + \text{LH to make Type 2} &\leq 585 \\ 3X_1 + 2X_2 &\leq 585 \end{aligned}$$

Once the pattern has been established, it becomes easy to write the labour constraints for Departments B and C. For Department B we must have $1.5X_1 + 5X_2 \leq$

500, and for Department C, we require that $4X_1 + 6X_2 \leq 900$. Once you have become used to problems like this, you may wish to write the constraints directly from the problem description without doing the table as an intermediate step. In summary the labour constraints are:

$$\begin{array}{lrcrcr}
\text{Dept. A Labour} & 3X_1 & + & 2X_2 & \leq & 585 \\
\text{Dept. B Labour} & 1.5X_1 & + & 5X_2 & \leq & 500 \\
\text{Dept. C Labour} & 4X_1 & + & 6X_2 & \leq & 900
\end{array}$$

Summary

The algebraic model needs to be summarized in one place. This summary consists of: the definition of the variables; the objective function; the words *subject to* followed by the constraints with their word descriptions; and the non-negativity restrictions written in one line at the end. For questions of this type on a test or examination in this course, just writing such a summary will suffice. Doing this we have:

$$\begin{array}{rl}
X_1 = & \text{the number of TPD of Type 1 cement made} \\
X_2 = & \text{the number of TPD of Type 2 cement made}
\end{array}$$

$$\begin{array}{rrcrcr}
\text{maximize} & 8X_1 & + & 10X_2 & & \\
\text{subject to} & & & & & \\
\text{Type 1 Sales} & X_1 & & & \geq & 40 \\
\text{Type 2 Sales} & & & X_2 & \geq & 30 \\
\text{Total Production} & X_1 & + & X_2 & \leq & 200 \\
\text{Dept. A Labour} & 3X_1 & + & 2X_2 & \leq & 585 \\
\text{Dept. B Labour} & 1.5X_1 & + & 5X_2 & \leq & 500 \\
\text{Dept. C Labour} & 4X_1 & + & 6X_2 & \leq & 900 \\
\text{non-negativity} & X_1 & , & X_2 & \geq & 0
\end{array}$$

After the optimal solution has been found, the fixed cost of \$1400 needs to be subtracted from the objective function value.

2.1.3 Plotting the Constraints

Introduction

From the Total Production constraint $X_1 + X_2 \leq 200$, we can see that a 200 by 200 grid is adequate for solving this problem. The convention is that the X_1 variable is on the horizontal axis, and the X_2 variable is on the vertical axis. A picture of the grid, with word descriptions on the axes, is shown in Figure 2.1. Though you will no doubt work with lined paper, here we suppress the printing of the grid lines to make the plotted lines easier to see.

We now need to plot the boundaries of the six constraints, and to do this we must find two points for each boundary line. Also, since all the constraints are inequalities, for each we must determine the direction of the arrow which indicates the inequality. The inequality divides the plane into two halves. On one side on the boundary, every point satisfies the inequality; this we will call the *true* side. On the other side of the inequality, no point satisfies the inequality; this we will call the *false* side.

In the common situation where both of the left-hand side coefficients are positive, and where the right-hand side coefficient is strictly positive, the origin (0,0) will be true for $\leq$ constraints and false for $\geq$ constraints. Let the top of the graph paper be considered "north". Therefore, for $\leq$ constraints the arrow which points to the true side will point down if the boundary is horizontal, point to the left if the boundary is vertical, and will point south-west for all other constraints. For $\geq$ constraints the arrow which points to the true side will point upwards if the boundary is horizontal, will point to the right if the boundary is vertical, and will point north-east for all other constraints. The situation where one of the left-hand side coefficients is negative is more complicated, and will be covered later in this chapter.

Three Easy Constraints

Three of the constraints are easy. The first one, $X_1 \geq 40$, is simply a *vertical* line through $X_1 = 40$, and the arrow points to the right (because the inequality makes the origin *false*). The second one, $X_2 \geq 30$, is a *horizontal* line through $X_2 = 30$, with the arrow pointing upwards. The third constraint, which is $X_1 + X_2 \leq 200$, passes through 200 on both axes. Since the origin is true, the arrow points towards the origin. The other three constraints require some calculations.

Department A Labour

The Department A Labour constraint is $3X_1 + 2X_2 \leq 585$. The boundary line of this constraint is given by the equation $3X_1 + 2X_2 = 585$. Setting $X_1 = 0$, we obtain $X_2 = 292.5$, which is off the grid. When this happens we try to find an interception point on either the right-hand side or the top boundary of the grid. In

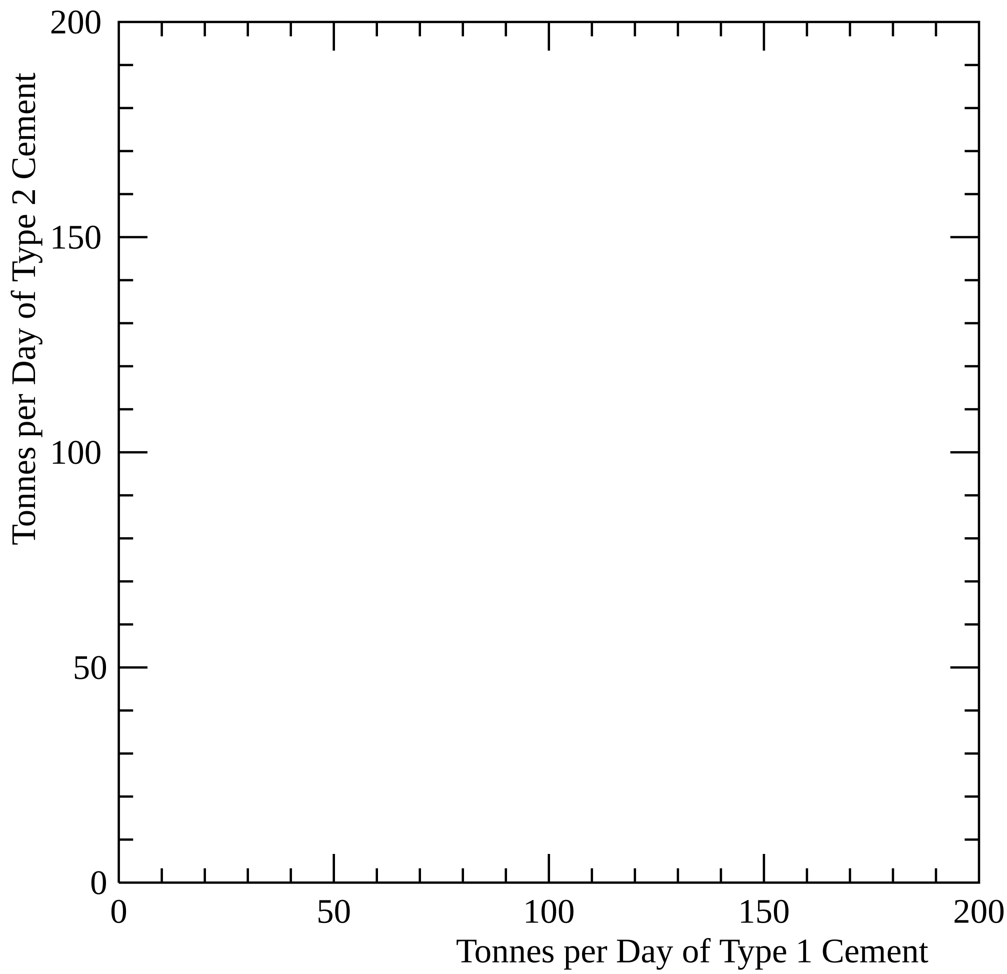


Figure 2.1: Cement Problem – Axes

this situation, we find the value of X_1 where the line crosses the top boundary, at which $X_2 = 200$. Hence we solve

$$\begin{aligned} 3X_1 + 2(200) &= 585 \\ 3X_1 + 400 &= 585 \\ 3X_1 &= 185 \\ X_1 &= 61.666... \end{aligned}$$

Hence the line passes through the point ($61\frac{2}{3}$,200). Now setting $X_2 = 0$, we obtain $X_1 = 195$, which is on the grid. Therefore the boundary of the Department A Labour constraint passes through the points ($61\frac{2}{3}$,200) and (195,0).

The origin is true for labour constraint A, so the arrow points towards the origin.

Departments B and C Labour

For Department B we require that $1.5X_1 + 5X_2 \leq 500$, whose boundary is given by $1.5X_1 + 5X_2 = 500$. Setting $X_1 = 0$, we obtain $X_2 = 100$, which is fine. Setting $X_2 = 0$ makes $X_1 = 333.333...$, which is off the grid. Therefore we set $X_1 = 200$ (the right-hand side of the grid), and solve to obtain $X_2 = 40$.

For Department C we require that $4X_1 + 6X_2 \leq 900$, whose boundary line is given by $4X_1 + 6X_2 = 900$. Setting $X_1 = 0$, we obtain $X_2 = 150$, which is fine. Setting $X_2 = 0$ makes $X_1 = 225$, which is off the grid. Therefore we set $X_1 = 200$ and solve to obtain $X_2 = 16.666....$

The origin is true for labour constraints B and C, so the arrow for each one points towards the origin.

Summary of Points for the Boundaries

In summary, the points for the boundary lines of the constraints are as follows:

Constraint	First Point	Second Point
Type 1 Sales	(40,0)	vertical
Type 2 Sales	(0,30)	horizontal
Total Production	(0,200)	(200,0)
Dept. A Labour	($61\frac{2}{3}$,200)	(195,0)
Dept. B Labour	(0,100)	(200,40)
Dept. C Labour	(0,150)	(200,$16\frac{2}{3}$)

The points can also be displayed with the algebraic model:

$X_1 =$ the number of TPD of Type 1 cement made
$X_2 =$ the number of TPD of Type 2 cement made

						First Point	Second Point
maximize	$8X_1$	$+$	$10X_2$				
subject to						First Point	Second Point
Type 1 Sales	X_1			$\geq$	40	$(40,0)$	vertical
Type 2 Sales			X_2	$\geq$	30	$(0,30)$	horizontal
Total Production	X_1	$+$	X_2	$\leq$	200	$(0,200)$	$(200,0)$
Dept. A Labour	$3X_1$	$+$	$2X_2$	$\leq$	585	$(61\frac{2}{3},200)$	$(195,0)$
Dept. B Labour	$1.5X_1$	$+$	$5X_2$	$\leq$	500	$(0,100)$	$(200,40)$
Dept. C Labour	$4X_1$	$+$	$6X_2$	$\leq$	900	$(0,150)$	$(200,16\frac{2}{3})$
non-negativity	X_1	,	X_2	$\geq$	0		

2.1.4 Finding the Feasible Region

All of the labour constraints have both coefficients positive on the left-hand side, so all the corresponding arrows point south-west towards the origin. A picture of all the constraints, showing the boundary lines and arrows which point to the true side, is shown in Figure 2.2. We must remember not to plot the points for the constraints backwards. For example, (0,100) lies 100 points *above* the origin, not 100 points to the right. The title of each constraint is written next to its boundary line. With these titles on the constraints, and with word descriptions on the axes, it makes the graph easy to understand.

We have not drawn arrows to explicitly indicate the two non-negativity restrictions, but of course these restrictions are present nevertheless. Considering all the constraints and the non-negativity restrictions, we find the feasible region. This region, which is labelled and highlighted, is shown in Figure 2.3.

2.1.5 Plotting a Trial Isovalue Line

At a Particular Objective Function Value

We now find a trial *isovalue* line, a line in which all points have the same objective function value. This being done, we then find the *optimal* isovalue line, a line parallel with the trial isovalue line which passes through the optimal solution.

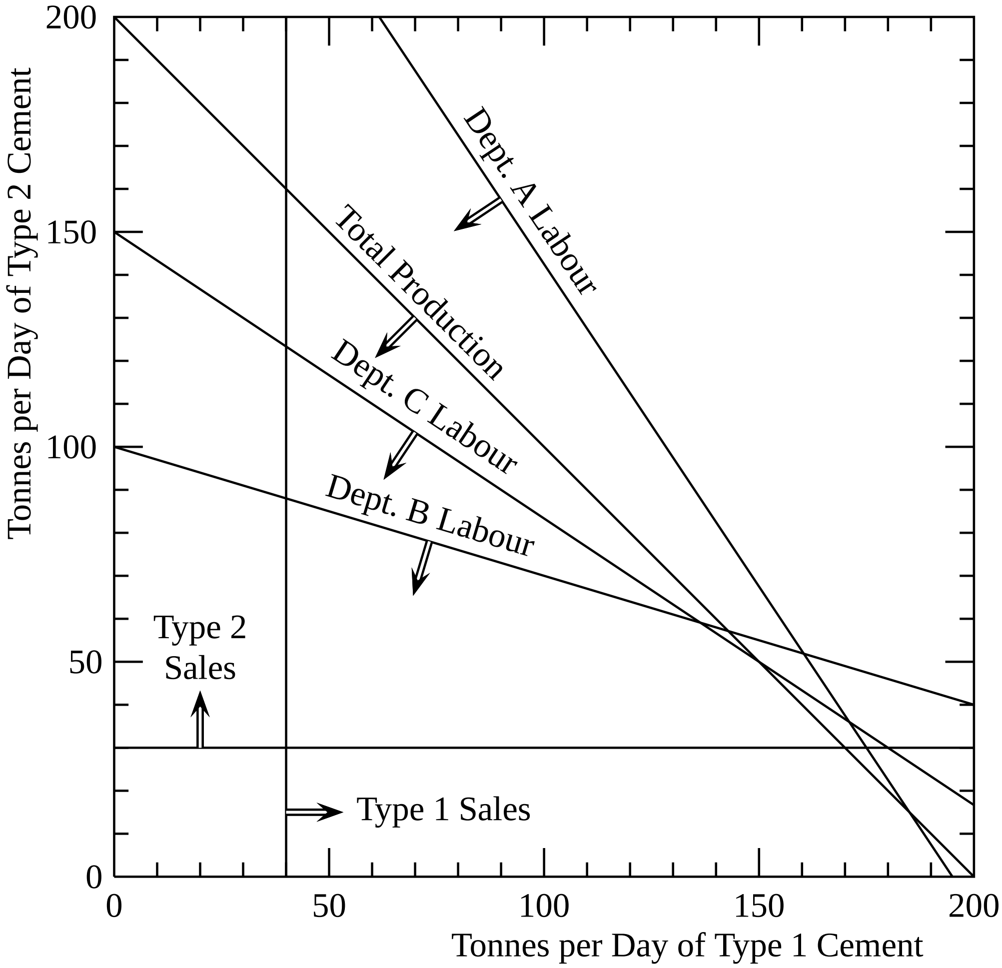


Figure 2.2: Cement Problem – Constraints

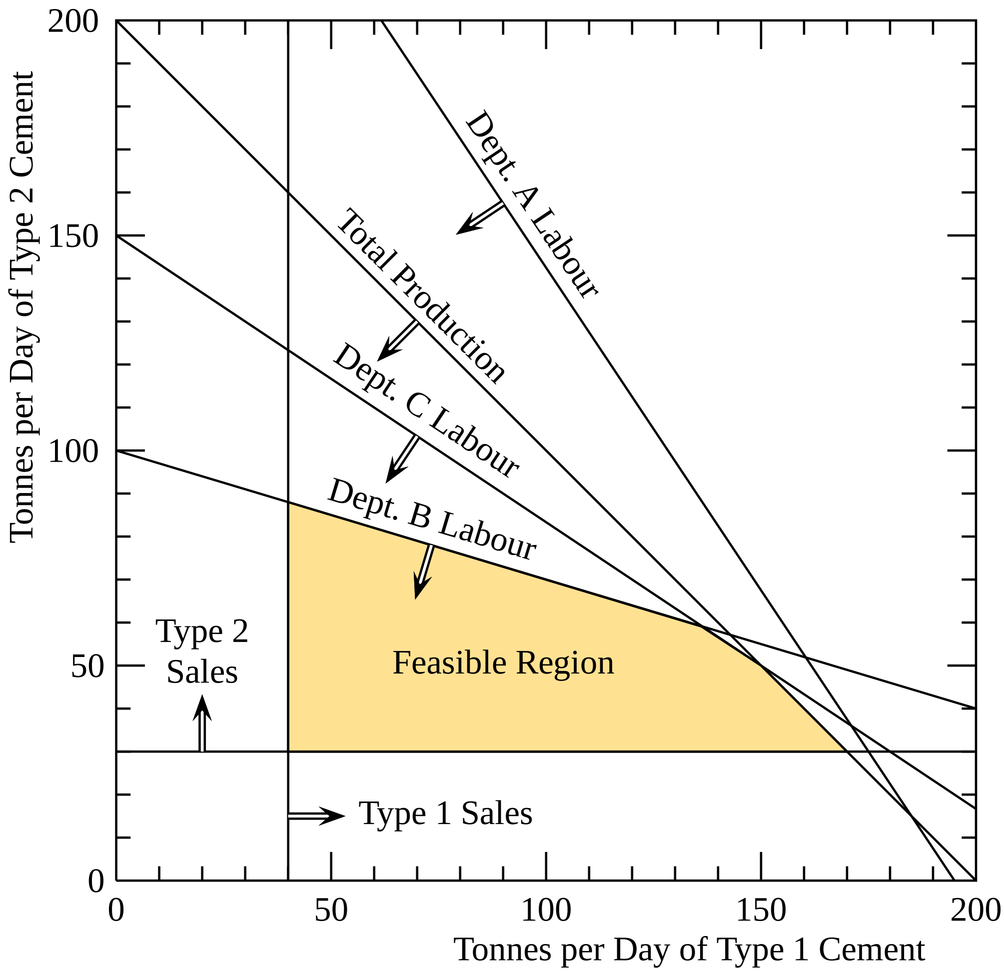


Figure 2.3: Cement Problem – Feasible Region

In general the objective function is of the form

$$\text{max or min } c_1X_1 + c_2X_2$$

Except when either $c_1 = 0$ or $c_2 = 0$ (which lead to horizontal and vertical isovalue lines respectively), we pick any value v (except 0) and solve

$$c_1X_1 + c_2X_2 = v$$

Using this equation we set each variable equal to 0 to obtain the intercepts on the axes. These two points define the isovalue line, which is indicated by drawing a dashed line between them.

For example, suppose we have $8X_1 + 10X_2$ in the objective function and wish to try $v = 200$, i.e. $8X_1 + 10X_2 = 200$. If $X_1 = 0$, then $10X_2 = 200$, and hence $X_2 = 20$. If $X_2 = 0$, then $8X_1 = 200$ and hence $X_1 = 25$. Therefore this particular isovalue line passes through $(X_1, X_2) = (0, 20)$ and (25,0), and we connect these two points with a dashed line.

An Easy Shortcut

However, while any non-zero value of v can be used, the special case where v is the product of c_1 and c_2 (where $c_1 \neq 0$ and $c_2 \neq 0$) leads to an easy shortcut:

$$c_1X_1 + c_2X_2 = c_1c_2$$

If $X_1 = 0$, then $c_2X_2 = c_1c_2$, and hence $X_2 = c_1$. Similarly, if $X_2 = 0$, then $X_1 = c_2$. Hence this line passes through the points $(0, c_1)$ and $(c_2, 0)$, i.e. the line goes from c_1 on the vertical axis to c_2 on the horizontal axis. In other words, the shortcut is this:

1. Plot the coefficient of X_1 on the *vertical* axis. 2. Plot the coefficient of X_2 on the *horizontal* axis.

As the two variables could be named differently, such as X and Y, a more general set of rules is:

1. Plot the coefficient of the horizontal variable on the *vertical* axis. 2. Plot the coefficient of the vertical variable on the *horizontal* axis.

These two points are then connected by a dashed line to represent a trial isovalue line.

Some Exceptions Of course, the shortcut may produce points that are too close to the origin to be able to draw the connecting line, in which case we need to multiply each intercept by a number greater than 1. At the other extreme, the shortcut may produce intercepts which are off the page, in which case we need to multiply each intercept by a number between 0 and 1.

The Cement Example For the example at hand, we seek to maximize $8X_1 + 10X_2$. Using the shortcut we obtain a *vertical* intercept of 8 and a *horizontal* intercept of 10. However, this does not help us much here, because (0,8) and (10,0) are in the bottom left-hand corner, so it's hard to draw the line between them. Hence we multiply each of these intercepts by a number greater than 1. For example, multiplying each intercept by 10 we obtain a vertical intercept of 80 and a horizontal intercept of 100. These points (0,80) and (100,0) are what we would have obtained if we had set $8X_1 + 10X_2$ to a trial value of $v = 800$, and then solved for the intercepts. Of course, there are an infinite number of trial isovalue lines, and any one of them would suffice.

Multiple Isovalue Lines In the picture shown in Figure 2.4, a family of isovalue lines is shown. For clarity, the constraints were removed, showing only the feasible region and the set of isovalue lines. Any one of these could be used as a trial isovalue line, though we will use the one for which OFV = 800. Note that the OFV increases as we move "north-east". We can see from this picture that the OFV of the optimal solution must be greater than 1600, but it must also be less than 1800, as no part of the corresponding isovalue line touches the feasible region. This picture has been drawn only to illustrate that multiple isovalue lines exist, and that there is an improving direction. Drawing this picture is not done as part of the solution process, because all we need is one trial isovalue line.

2.1.6 Finding the Optimal Solution

We now find a line parallel with the trial isovalue line, which just passes through the boundary[2] of the feasible region such that the objective function value is maximized. A convenient means of doing this is to use a *rolling ruler*, but a triangle

[2] Usually the optimal solution occurs at a corner of the feasible region, but when there is multiple optimality an entire edge of the feasible region will be optimal. In any case, no part of the optimal isovalue line will appear inside the feasible region.

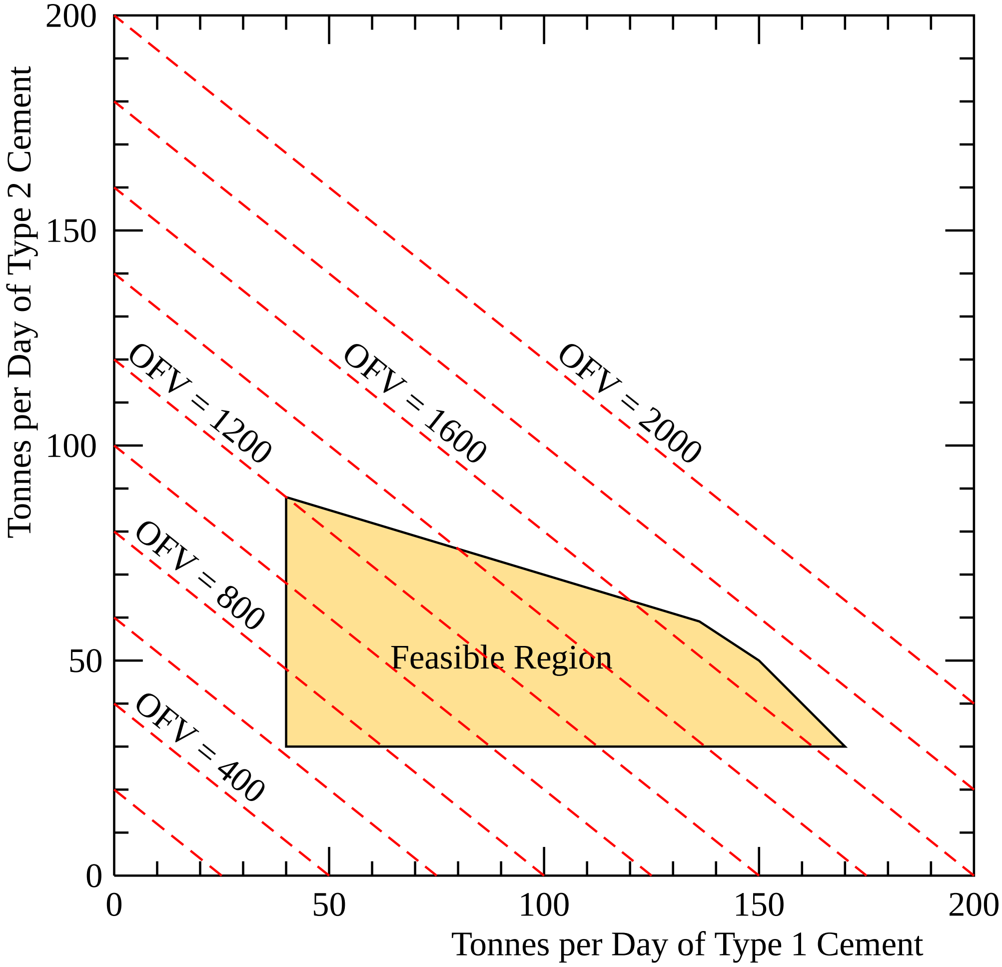


Figure 2.4: Cement Problem – Feasible Region and a set of Isovalue Lines

moved along a straightedge will work too. This optimal isovalue line is also drawn on the graph (again, as a dashed line), and the optimal solution is identified.

A constraint is said to be *binding* if its boundary passes through the optimal solution.

From the graph we can see that the binding constraints for this example are the ones for (i) Total Production and (ii) Department C Labour, and that the optimal solution appears to be at about $X_1 = 150$ and $X_2 = 50$. A picture of this is shown in Figure 2.5.

2.1.7 Finding the Exact Solution

Using Algebra

By taking the boundaries of the two binding constraints, we can obtain the solution exactly:

$$\begin{array}{rrcrcr}
\text{Total Production} & X_1 & + & X_2 & = & 200 \\
\text{Dept. C Labour} & 4X_1 & + & 6X_2 & = & 900 \\
\\
 & 6X_1 & + & 6X_2 & = & 1200 \\
 & 4X_1 & + & 6X_2 & = & 900 \\
\\
 & 2X_1 & + & 0X_2 & = & 300 \\
\\
 & X_1 & & & = & 150
\end{array}$$

By substituting $X_1 = 150$ into $X_1 + X_2 = 200$, we obtain $X_2 = 50$. The optimal mathematical solution is $X_1^* = 150$ and $X_2^* = 50$. (Asterisks are used to indicate optimality.)

Using Matrix Operations in Excel (Optional)

Alternatively, we could solve the equations using Excel. Beginning with

$$\begin{array}{rrcrcr}
\text{Total Production} & X_1 & + & X_2 & = & 200 \\
\text{Dept. C Labour} & 4X_1 & + & 6X_2 & = & 900
\end{array}$$

we convert these equations to matrix form:

$$\begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 200 \\ 900 \end{bmatrix}$$

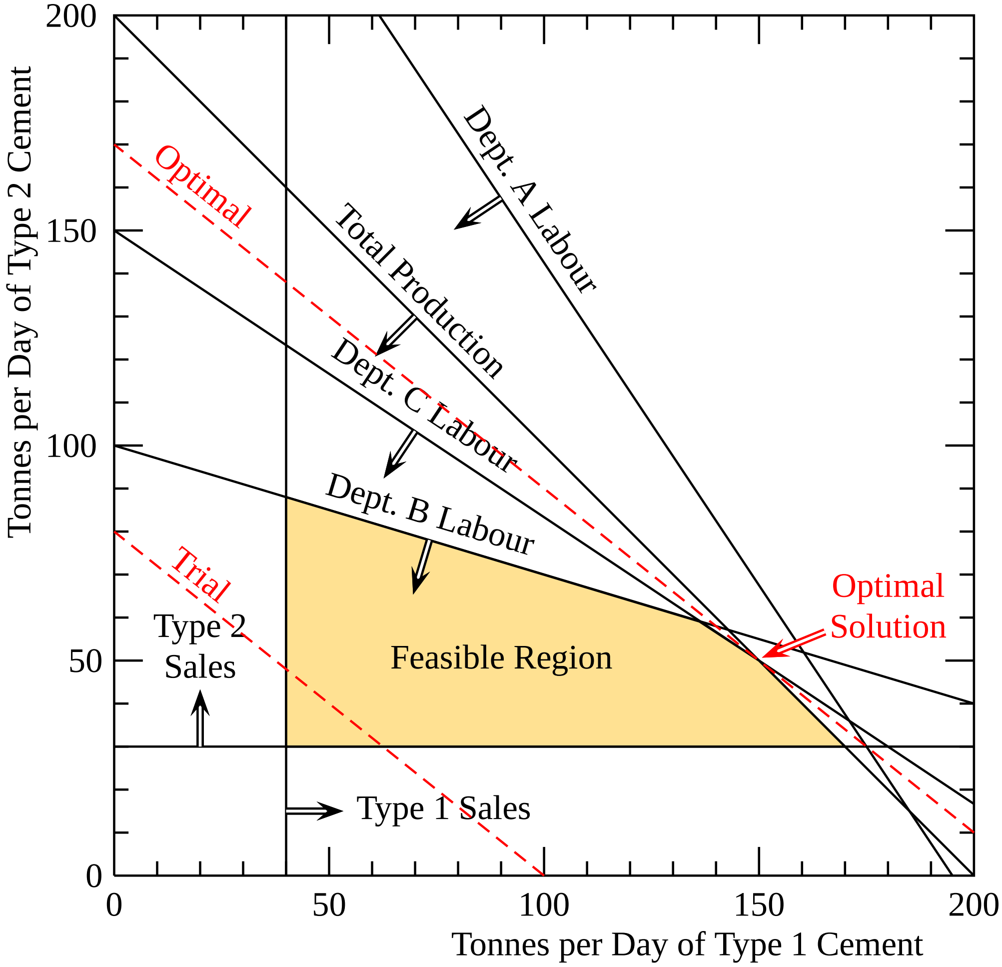


Figure 2.5: Cement Problem – Optimal Solution

Using the Excel MINVERSE function to perform the matrix inversion we obtain:

$$\begin{bmatrix} 1 & 1 \\ 4 & 6 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & -0.5 \\ -2 & 0.5 \end{bmatrix}$$

Using MMULT to multiply the inverse by the right-hand side values, we obtain:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 150 \\ 50 \end{bmatrix}$$

The unique solution is $X_1 = 150$, and $X_2 = 50$.

The OFV

The objective function value at the point of optimality is

$$\begin{aligned} \text{OFV}^* &= 8X_1^* + 10X_2^* \\ &= 8(150) + 10(50) \\ &= 1200 + 500 \\ &= 1700 \end{aligned}$$

Going back to the original problem, the solution expressed in managerial terms is:

Recommendation

The cement plant should produce 150 Tonnes per day of Type 1 cement, and 50 Tonnes per day of Type 2 cement, for a contribution to profit of \$1700 per day. After deducting the \$1400 daily fixed costs, the net profit is \$300 per day.

2.2 Extensions

2.2.1 General Form

To be considered a linear optimization model, we must have a linear objective function and linear constraints. By *linear* we mean that in each expression:

1. A variable cannot be multiplied by another variable (for example, we cannot have something like $7X_1X_2$).

2. Every variable is multiplied by a number (which can be positive, zero, or negative) only (for example, we cannot have something like $5\sqrt{X_1}$).

3. No uncertainty is permitted.

4. The variables must be able to take on real (as opposed to integer) values.[3]

We can, however, make the model with *minimization* rather than maximization as the objective, and the constraints can be equalities as well as the more usual $\leq$ and $\geq$ inequalities. The number on the right-hand side can be zero (considered below). The non-negativity restrictions are almost always present, but these can be removed when it is appropriate to do so. The number of variables and the number of constraints is theoretically unlimited, but the software to solve the model will come with limitations. There is no problem in practice solving models with thousands of variables, and indeed solving models with millions of variables is sometimes done.

2.2.2 A Right-Hand Side Value of 0

Introduction

Here we consider the case where the number on the right-hand side of a constraint is 0. In the next section, we will see the modeling of such a constraint. For now, we are just interested in learning how to plot such a constraint on a graph. In a two-variable problem, when such constraints appear one of the two variables will have a negative coefficient. Here are some examples:

$$\begin{array}{lrcl} (1) & 3X - 8Y & \geq & 0 \\ (2) & 0.6X_1 - 0.4X_2 & \leq & 0 \\ (3) & -6L + 3S & \geq & 0 \\ (4) & -0.4X_1 + 0.2X_2 & \leq & 0 \end{array}$$

In all of the above, we begin by plotting the boundary of each constraint, which is an equality. We will return to the inequality when determining the direction of the arrow. The equations for the boundaries are:

$$\begin{array}{lrcl} (1) & 3X - 8Y & = & 0 \\ (2) & 0.6X_1 - 0.4X_2 & = & 0 \\ (3) & -6L + 3S & = & 0 \\ (4) & -0.4X_1 + 0.2X_2 & = & 0 \end{array}$$

[3] Later, when we look at Integer Models, we will relax this fourth assumption.

In any of the above, we see that if one variable is set equal to 0, then the other variable will also be 0. Hence anytime we have a constraint with a 0 on the right-hand side, the constraint will pass through (0,0). We need to find another distinct point on the boundary line.

Finding a Second Point on the Line

In constraint (1), we see that if we make $X = 8$, then Y will be 3, since $3(8) - 8(3) = 0$. Hence, in addition to the point (0,0), this line will pass through (8,3). If the graph paper is say 300 by 300, the point (8,3) will be so close to (0,0) that it would be difficult to draw the line accurately. What we need to do therefore is multiply both numbers by any positive number, as long as we do not go outside the graph.[4] For example, multiplying by 10 would give us the point (80,30). but other possibilities would be to multiply by 20 to give (160,60), or multiply by 30 to give (240,120).

In general, we can determine the coordinates of the second point by switching the absolute value of the two coefficients. Hence, constraint (2), whose coefficients are 0.6 and -0.4, will pass through (0.4,0.6). Or, we could scale this point up by multiplying by 100 to obtain (40,60), or multiplying by 200 to obtain (80,120), or even multiplying by 500 to obtain (200,300).

The coefficients of (3) are -6 and 3, hence (3) passes through (3,6), or any multiple such as (150,300). The coefficients of (4) are -0.4 and 0.2, hence (4) passes through (0.2,0.4), or any multiple such as (100,200).

The Direction of the Arrow

When the boundary of a constraint does not pass through (0,0) (which is what happens most of the time), we would choose (0,0) as trial point. If the constraint is true for (0,0), the arrow points towards the origin; if false, it points away from the origin.

The problem now is that the boundary of a constraint whose right-hand side value is 0 does pass through the origin, so (0,0) cannot be used as a trial point. We can pick any point which is not on the line and use it as a trial point. For example, we could pick (100,0). We substitute this point into the left-hand side of the constraint. If the constraint is true at this value, then the arrow points to the

[4] We can only do this scalar multiplication because the line goes through the origin; this does not work in other situations.

side of the boundary line which contains (100,0); if false, the arrow will point the other way. Now we test each constraint using this particular trial point.

Constraint (1) We require that $3X - 8Y \geq 0$. At the trial point (100,0) we obtain $3(100) - 8(0) = 300 \geq 0$, hence (100,0) is true, and therefore all points south-east of the boundary are true. The arrow points south-east.

Constraint (2) We require that $0.6X_1 - 0.4X_2 \leq 0$. At the trial point (100,0) we obtain $0.6(100) - 0.4(0) = 60 \nleq 0$, hence (100,0) is false, and therefore all points south-west of the boundary are false. The true points lie north-west of the boundary; the arrow points north-west.

Constraint (3) We require that $-6L + 3S \geq 0$. At the trial point (100,0) we obtain $-6(100) + 3(0) = -600 \ngeq 0$, hence (100,0) is false, and therefore all points south-west of the boundary are false. The true points lie north-west of the boundary; the arrow points north-west.

Constraint (4) We require that $-0.4X_1 + 0.2X_2 \leq 0$. At the trial point (100,0) we obtain $-0.4(100) + 0.2(0) = -40 \leq 0$, hence (100,0) is true, and therefore all points south-east of the boundary are true. The arrow points south-east.

2.3 Cement Model with a Proportion Constraint

2.3.1 A Revised Model

Calling the original cement plant model (a), we now consider a modification, which we denote as (b).

Suppose now that the cement model is as it was before, but now the amount of Type 1 cement production cannot exceed two-thirds of the total amount produced. This is not necessarily two-thirds of 200 TPD, because we do not know in advance that this constraint will be binding. There are two approaches which can be used:

1. Keep the model with two variables, recognizing that the total amount produced is $X_1 + X_2$. This approach allows for the model's solution using the graphical method, but the two-thirds figure will no longer be transparent.

2. Let X_3 represent the total amount produced. This approach preserves the two-thirds figure, but to find the solution we will need to use a computer.

Here we use the first approach; the second approach appears on page 80.

The amount of Type 1 cement cannot exceed 2/3 of the combined production of Type 1 and Type 2 cement, therefore:

$$\begin{aligned} X_1 &\leq \tfrac{2}{3}(X_1+X_2) \\ 3X_1 &\leq 2X_1+2X_2 \\ X_1-2X_2 &\leq 0 \end{aligned}$$

In the second line, we cross-multiplied by 3, to avoid the repeating decimal.[5]

Model (b) with this new constraint added is:

$X_1 =$ the number of TPD of Type 1 cement made
$X_2 =$ the number of TPD of Type 2 cement made

$$\begin{array}{rrcrcr} \text{maximize} & 8X_1 & + & 10X_2 & & \\ \text{subject to} & & & & & \\ \text{Type 1 Sales} & X_1 & & & \geq & 40 \\ \text{Type 2 Sales} & & & X_2 & \geq & 30 \\ \text{Total Production} & X_1 & + & X_2 & \leq & 200 \\ \text{Dept. A Labour} & 3X_1 & + & 2X_2 & \leq & 585 \\ \text{Dept. B Labour} & 1.5X_1 & + & 5X_2 & \leq & 500 \\ \text{Dept. C Labour} & 4X_1 & + & 6X_2 & \leq & 900 \\ \text{Part (b) Proportion} & X_1 & - & 2X_2 & \leq & 0 \\ \text{non-negativity} & X_1 & , & X_2 & \geq & 0 \end{array}$$

2.3.2 A Right-Hand-Side Value of 0

What's new here is that we now must plot a constraint whose right-hand-side (RHS) value is 0. As mentioned earlier, the boundary of *any* constraint with a 0 on the right-hand-side will pass through the origin. A second point is obtained by switching the absolute value of the coefficients; the boundary passes through

[5] If the fraction had been something like $\frac{3}{4}$, we would have used the decimal equivalent 0.75; there would be no need for cross-multiplication.

(2,1). To obtain a better point for drawing the line, we multiply by 100 to obtain the point (200,100). The boundary passes through $(X_1, X_2) = (0,0)$ and (200,100).

On any constraint which has a negative number on the left-hand side, and especially for one where the right-hand side value is 0, a great deal of care must be taken to make sure that the arrow is drawn in the correct direction. We must pick a point which is not on the line, such as (100,0). Substituting $X_1 = 100$ and $X_2 = 0$ into $X_1 - 2X_2 \leq 0$ gives us $100 - 2(0) = 100 \not\leq 0$, and so the point (100,0) is false. All points on this (100,0) side of the boundary line are false. Therefore the arrow points away from the point (100,0); i.e. the arrow points north-west.

2.3.3 The Feasible Region

Superimposing this constraint on the existing solution produces an altered feasible region; a part of the former feasible region has now become infeasible. In Figure 2.6, the new feasible region is shown in gold, the now infeasible part of the former feasible region is shown in light blue, and the old and new optimal solutions are shown.

The binding constraints now are the Department B Labour constraint and the Proportion constraint, with the optimal solution occuring at $X_1 \approx 125$ and $X_2 \approx 60$. (The symbol $\approx$ means "approximately".)

2.3.4 Finding the Exact Solution

Using Algebra At the boundaries of these constraints we obtain the exact solution:

$$\begin{array}{rrcrcr}
\text{Dept. B Labour} & 1.5X_1 & + & 5X_2 & = & 500 \\
\text{Proportion} & X_1 & - & 2X_2 & = & 0 \\
\\
 & 3X_1 & + & 10X_2 & = & 1000 \\
 & 5X_1 & - & 10X_2 & = & 0 \\
\\
 & 8X_1 & + & 0X_2 & = & 1000 \\
\\
 & X_1 & & & = & 125
\end{array}$$

By substituting $X_1 = 125$ into $X_1 - 2X_2 = 0$, we obtain $X_2 = 62.5$. The optimal mathematical solution for the altered model is $X_1^* = 125$ and $X_2^* = 62.5$.

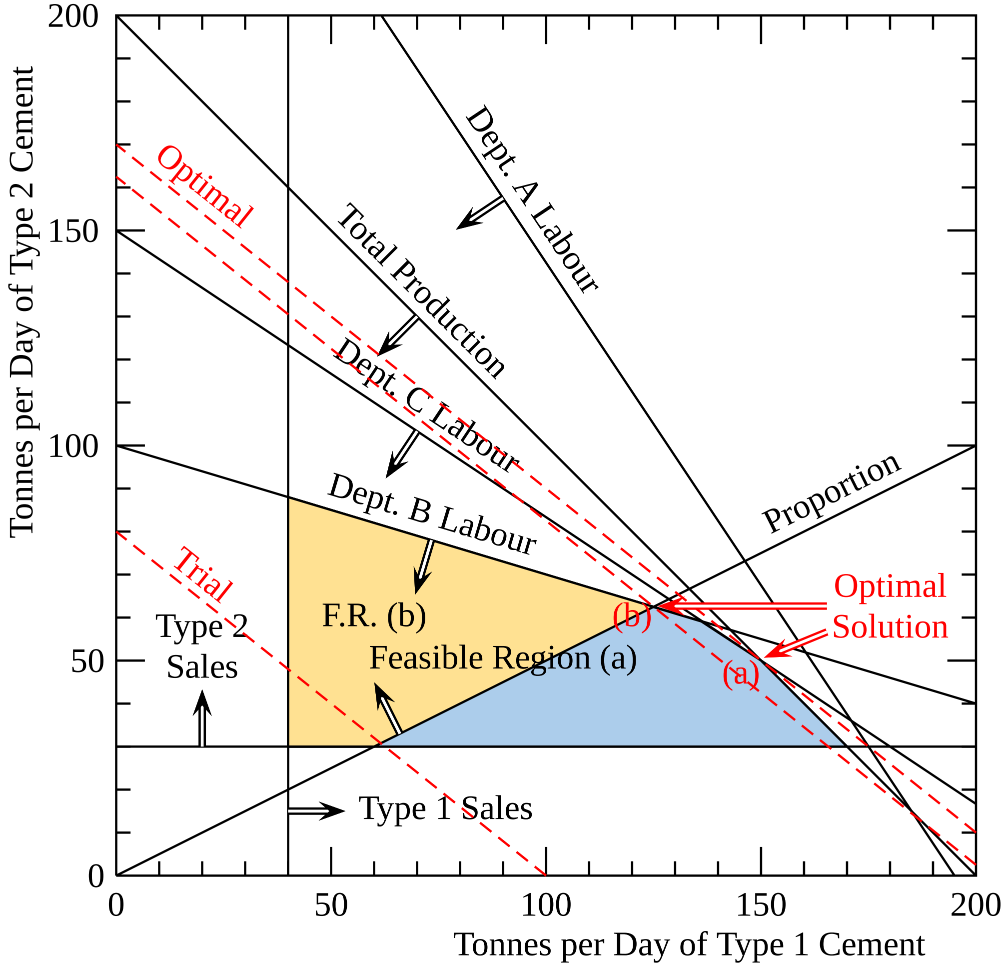


Figure 2.6: Cement Problem – Altered Optimal Solution

Using Matrix Operations in Excel (Optional) Alternatively, we could solve the equations using Excel. Beginning with

$$\begin{array}{rrcrcr} \text{Dept. B Labour} & 1.5X_1 & + & 5X_2 & = & 500 \\ \text{Proportion} & X_1 & - & 2X_2 & = & 0 \end{array}$$

we convert these equations to matrix form:

$$\begin{bmatrix} 1.5 & 5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 500 \\ 0 \end{bmatrix}$$

Using the Excel MINVERSE function to perform the matrix inversion we obtain:

$$\begin{bmatrix} 1.5 & 5 \\ 1 & -2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & 0.625 \\ 0.125 & -0.1875 \end{bmatrix}$$

Using MMULT to multiply the inverse by the right-hand side values, we obtain:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 125 \\ 62.5 \end{bmatrix}$$

The unique solution is $X_1 = 125$, and $X_2 = 62.5$.

The OFV The objective function value at the point of optimality is

$$\begin{aligned} \text{OFV}^* &= 8X_1^* + 10X_2^* \\ &= 8(125) + 10(62.5) \\ &= 1000 + 625 \\ &= 1625 \end{aligned}$$

Going back to the original problem, the solution expressed in managerial terms is:

Recommendation

With the added requirement that the level of Type 1 production cannot exceed two-thirds of the total production, the cement plant should produce 125 Tonnes per day of Type 1 cement, and 62.5 Tonnes per day of Type 2 cement, for a contribution to profit of \$1625 per day. After deducting fixed costs of \$1400 per day, the net daily profit will be \$225.

Comment

The initial model with its six constraints leads to a solution which creates a (daily contribution to) profit of $1700. Then, after adding a seventh constraint, the profit fell to $1625. Whenever a constraint is added, the profit can at best stay the same, and often it will fall. In general, adding another constraint (or making an existing one more stringent) can at best keep the OFV the same, otherwise it will be impaired. By impaired, we mean that the OFV will decrease if the objective is maximization, and will increase if the objective is minimization. Note that while the profit went down, only one of the variables did. The Type 1 cement production decreased from 150 to 125 TPD, but the Type 2 cement production increased from 50 to 62.5 TPD.

2.4 Example – Diet Problem

2.4.1 Problem Description

This example is made to illustrate linear optimization. Don't take it as nutritional advice. A real diet shouldn't contain only these two items.

A twenty-two year old student lives on a diet of double hamburgers and orange juice. To make a double hamburger (bun, two patties of beef, and condiments) costs about $1.25, and a serving (249 g) of unsweetened orange juice costs about $0.32. She wants to minimize her daily cost of buying these things, but she has decided to make sure that she obtains the recommended daily intake of all vitamins and minerals. To keep the problem simple, she wants the protein, iron, and Vitamin C to meet or exceed the recommended amounts for a woman of her age, and to restrict the amount of iron from hamburgers to be no more than 90% of her total iron intake.

A search on the web [6] gives the amounts of the three nutrients per serving of food:

Nutrient	Double Hamburger (per 215 g sandwich)	Orange Juice (per 249 g serving)
Protein (g)	31.820	1.469
Iron (mg)	5.547	1.096
Vitamin C (mg)	1.075	85.656

[6] The website https://www.nutrition.gov/ was used to obtain this information, but the data might now be superseded.

Another search [7] reveals that a woman in the age group 19-24 needs 46 g of Protein, 15 mg of Iron, and 60 mg of Vitamin C per day.[8]

Her objective is to minimize the cost of her diet.

2.4.2 Formulation

Variables and the Objective Function

In order to determine how much she is spending on her daily diet, we need to know the amounts consumed each day of hamburgers and orange juice. We must therefore have the following two decision variables:

$X_1 =$ the number of double hamburgers eaten each day
$X_2 =$ the number of servings of orange juice drunk each day

Each double hamburger costs \$1.25, and each serving of orange juice costs \$0.32, hence the objective function is:

$$\text{minimize} \;\; 1.25X_1 + 0.32X_2$$

The Constraints

We begin with the first three constraints, one for each of three nutrients. The purpose of these three constraints is to ensure that the recommended daily intake (RDI) is met.

The Protein Constraint For any constraint the units must match up on the left-hand and right-hand sides. The amount of protein consumed each day is:

$$\begin{aligned}
\text{total protein} \;&=\; \text{protein from hamburgers} + \text{protein from orange juice} \\
&=\; 31.820 \text{ grams/hamburger } \times X_1 \text{ hamburgers } + \\
&\;\; 1.469 \text{ grams/serving of orange juice } \times X_2 \text{ servings of orange juice} \\
&=\; 31.820X_1 \text{ grams } + 1.469X_2 \text{ grams}
\end{aligned}$$

[7] The original source is the Food and Nutrition Board - National Academy of Sciences, 1998 (University of California, Davis).

[8] Nutritional advice is constantly changing, so this data is used only for the purpose of illustrating the use of linear optimization in this context. See, for example, https://www.canada.ca/en/health-canada/services/food-nutrition.html for current advice.

Her RDI is for 46 grams of protein. To ensure that she obtains at least this amount we use a $\geq$ constraint:

$$31.820X_1 \text{ grams } + 1.469X_2 \text{ grams } \geq 46 \text{ grams}$$

With the sameness of the units on both sides, we can remove the word *grams* to obtain:

$$31.820X_1 + 1.469X_2 \geq 46$$

The Iron and Vitamin C Constraints The iron and Vitamin C constraints are in milligrams rather than grams, but the idea is the same. We obtain units of milligrams on both sides of the inequality, and hence the word *milligrams* can be dropped from both sides. The constraint for the iron requirement is:

$$5.547X_1 + 1.096X_2 \geq 15$$

The constraint for the Vitamin C requirement is:

$$1.075X_1 + 85.656X_2 \geq 60$$

The Iron Proportion Constraint Now we must restrict the iron from hamburgers to be no more than 90% of the total iron consumed. Here we recognize that the total iron consumed is $5.547X_1 + 1.096X_2$. An alternate approach which defines a new variable appears on page 81.

We create the Iron Proportion constraint as follows:

$$\begin{aligned} 5.547X_1 &\leq 0.9(5.547X_1 + 1.096X_2) \\ 0.1(5.547X_1) - 0.9(1.096X_2) &\leq 0 \\ 0.5547X_1 - 0.9864X_2 &\leq 0 \end{aligned}$$

The entire diet model is::

X_1 = the number of double hamburgers eaten each day
X_2 = the number of servings of orange juice drunk each day

$$
\begin{array}{rrcrcr}
\text{minimize} & 1.25X_1 & + & 0.32X_2 & & \\
\text{subject to} & & & & & \\
\text{Protein RDI} & 31.820X_1 & + & 1.469X_2 & \geq & 46 \\
\text{Iron RDI} & 5.547X_1 & + & 1.096X_2 & \geq & 15 \\
\text{Vitamin C RDI} & 1.075X_1 & + & 85.656X_2 & \geq & 60 \\
\text{Iron Proportion} & 0.5547X_1 & - & 0.9864X_2 & \leq & 0 \\
\text{non-negativity} & X_1 & , & X_2 & \geq & 0
\end{array}
$$

2.4.3 Plotting the Constraints

A Scale for the Graph

To establish a reasonable scale for the graph, we can think of the context from which the model came. Suppose that she eats three meals a day, each being a double hamburger and a serving of orange juice. Mathematically, this would imply that $X_1 = 3$, and $X_2 = 3$. By plugging these values into the four constraints, we can see that this solution is feasible. Since we are trying to *minimize* the cost, the solution must be less than 3 for one of the two variables, and we can hope that it will be less than 3 for both of them. If the grid from (0,0) to (3,3) turns out to be too small, we can always expand it later.

Boundary Points

We try to find where the boundary of every constraint intercepts the axes. When this yields a point outside the grid, we find the intercept on the right-hand side ($X_1 = 3$) or top ($X_2 = 3$) boundary instead. For example, the boundary of the Protein RDI constraint is

$$31.820X_1 + 1.469X_2 = 46$$

Setting $X_1 = 0$ causes X_2 to be off the 3 by 3 grid. Hence we set $X_2 = 3$, and solve $31.820X_1 + 1.469(3) = 46$, obtaining $X_1 \approx 1.307$. Setting $X_2 = 0$ causes X_1 to be about 1.446, which is on the grid. Hence the two points for this constraint are (1.307,3) and (1.446,0). Doing this for every constraint we obtain:

Constraint	First Point	Second Point
Protein RDI	(1.307,3)	(1.446,0)
Iron RDI	(2.111,3)	(2.704,0)
Vitamin C RDI	(0,0.7005)	(3,0.6628)
Iron Proportion	(0,0)	(3,1.687)

Alternatively, this data can be written on the algebraic model:

$$X_1 = \text{the number of double hamburgers eaten each day}$$
$$X_2 = \text{the number of servings of orange juice drunk each day}$$

$$\begin{array}{rrcrcrcc}
\text{minimize} & 1.25X_1 & + & 0.32X_2 & & & & \\
\text{subject to} & & & & & & \text{First Point} & \text{Second Point} \\
\text{Protein RDI} & 31.820X_1 & + & 1.469X_2 & \geq & 46 & (1.307,3) & (1.446,0) \\
\text{Iron RDI} & 5.547X_1 & + & 1.096X_2 & \geq & 15 & (2.111,3) & (2.704,0) \\
\text{Vitamin C RDI} & 1.075X_1 & + & 85.656X_2 & \geq & 60 & (0,0.7005) & (3,0.6628) \\
\text{Iron Proportion} & 0.5547X_1 & - & 0.9864X_2 & \leq & 0 & (0,0) & (3,1.687) \\
\text{non-negativity} & X_1 & , & X_2 & \geq & 0 & &
\end{array}$$

Direction of the Arrows

Because the origin is *false* for each of the first three constraints, all three arrows point away from the origin. The fourth constraint passes through the origin, so we test a point which is not on the constraint boundary, such as (0,2). This point is *true* with respect to the inequality, so the arrow points toward this point, i.e. upwards and to the left. These four constraints, along with their arrows and word descriptions, are shown in Figure 2.7.

2.4.4 Feasible Region, Isovalue Lines, and the Optimal Solution

We now find and highlight the feasible region. In this example, the feasible region is of infinite size, but it is clipped by the boundaries of the grid. Plotting the trial isovalue line is quite easy in this situation. The objective function is to minimize $1.25X_1 + 0.32X_2$, so we try the shortcut of plotting 1.25 on the *vertical* axis and 0.32 on the *horizontal* axis, and connect them with a dashed line. We then move

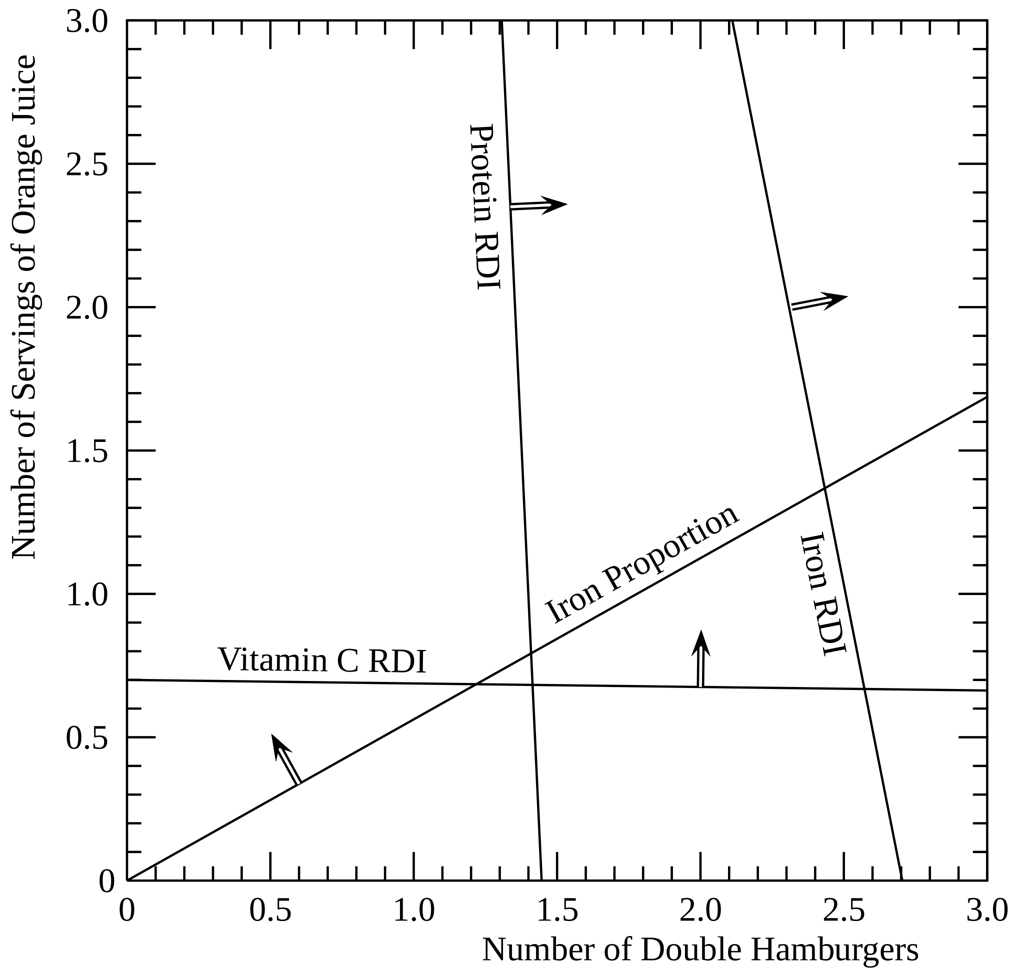


Figure 2.7: Diet Problem – Constraints

a rolling ruler over to the feasible region, stopping at the corner where the boundaries of the Iron RDI constraint and the Iron Proportion constraint intercept. This is shown in Figure 2.8. We can see that the optimal solution lies at about 2.4 double hamburgers per day, and 1.4 servings of orange juice per day.

2.4.5 Finding the Exact Solution

Using Algebra

To find the exact solution, we find the interception point of the boundaries of the Iron RDI and Iron Proportion constraints.

$$
\begin{array}{rrcrcr}
\text{Iron RDI} & 5.547X_1 & + & 1.096X_2 & = & 15 \\
\text{Iron Proportion} & 0.5547X_1 & - & 0.9864X_2 & = & 0 \\
\\
 & 5.547X_1 & + & 1.096X_2 & = & 15 \\
 & 5.547X_1 & - & 9.864X_2 & = & 0 \\
\\
 & 0X_1 & + & 10.96X_2 & = & 15 \\
\\
 & & & X_2 & \approx & 1.3686
\end{array}
$$

By substituting this value into either of the original constraints, we obtain $X_1 \approx 2.4337$.

Using Matrix Operations in Excel (Optional)

Alternatively, we could solve the equations using Excel. Beginning with

$$
\begin{array}{rrcrcr}
\text{Iron RDI} & 5.547X_1 & + & 1.096X_2 & = & 15 \\
\text{Iron Proportion} & 0.5547X_1 & - & 0.9864X_2 & = & 0
\end{array}
$$

we convert these equations to matrix form:

$$
\begin{bmatrix} 5.547 & 1.096 \\ 0.5547 & -0.9864 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 15 \\ 0 \end{bmatrix}
$$

Using the Excel MINVERSE function to perform the matrix inversion, and then using MMULT to multiply the inverse by the right-hand side values, we obtain:

$$
\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 2.433747972 \\ 1.368613139 \end{bmatrix}
$$

The unique solution is $X_1 = 2.433747972$, and $X_2 = 1.368613139$.

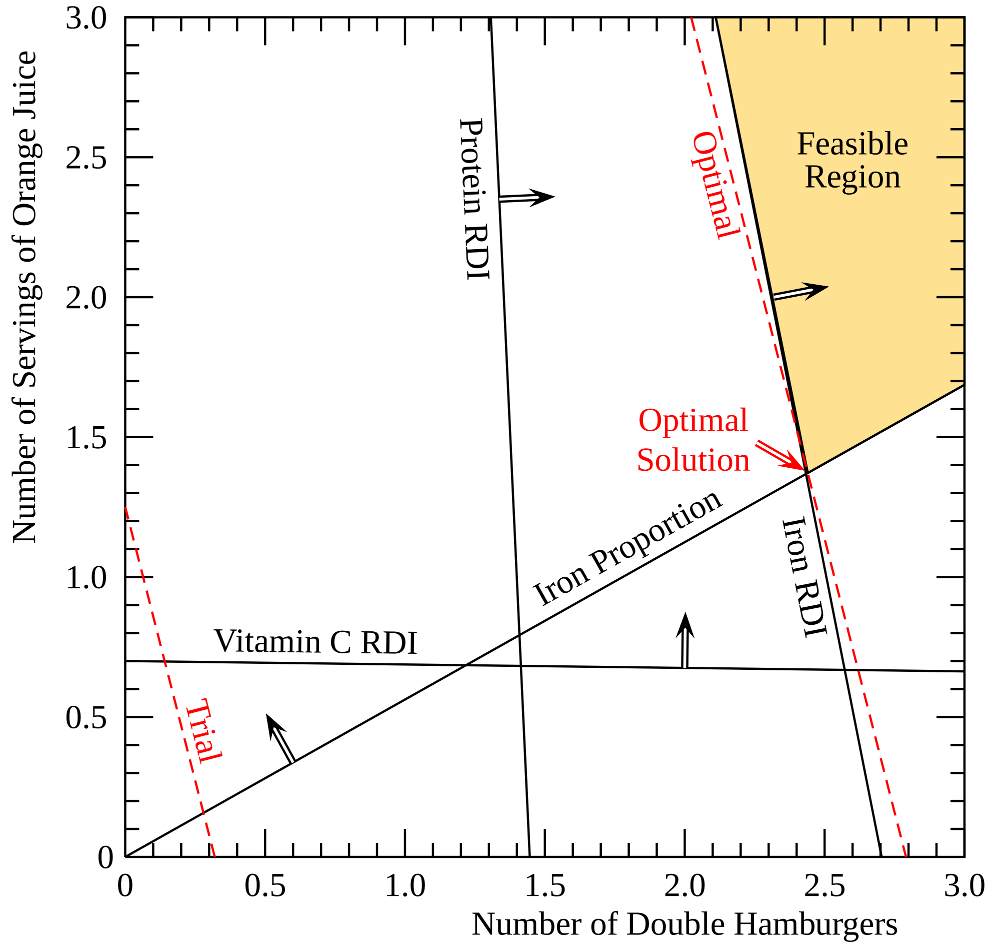


Figure 2.8: Diet Problem – Optimal Solution

The OFV

Using the exact values from Excel, the objective function value is:

$$\$1.25(2.433747972) + \$0.32(1.368613139) = \$3.48014 \approx \$3.48.$$

The question now arises as to whether we should recommend values for the variables which are not integer. To answer this question we need to consider the context of the problem. The orange juice is not a problem, because if we want 1.3686 servings of 249 g each, all we have to do is make two servings of 249/1.3686 $\approx$ 181.9 g each. The hamburgers are more of a problem, however, since it's hard to cook 0.4337 of a burger. However, for both the hamburgers and the orange juice, we can interpret the DRI for each nutrient as an average to be obtained over a period of time. For example, suppose that she eats two hamburgers and drinks one serving of orange juice on one day, and then eats three hamburgers and drinks two servings of orange juice on the next, and repeats this cycle. She would average 2.5 double hamburgers and 1.5 servings of orange juice over time. This would certainly meet the requirements of the DRI constraints ($2.5 > 2.4337$, and $1.5 > 1.3686$), and in the Iron Proportion constraint we have:

$$0.5547(2.5) - 0.9864(1.5) = -0.09285 < 0$$

Hence (2.5,1.5) is a feasible solution. The average daily cost is

$$\$1.25(2.5) + \$0.32(1.5) = \$3.605$$

which is about 12.5 cents higher than the theoretical optimal solution. We are now ready to make a recommendation.

Recommendation

Based on a self-imposed diet of double hamburgers and orange juice, and considering only the four stated nutritional requirements, a near-optimal solution can be implemented by eating two hamburgers and drinking one serving of orange juice on one day, and then eating three hamburgers and drinking two servings of orange juice on the next, and repeating this cycle. This gives an average daily cost of \$3.605.

2.5 Solution by Using the Excel Solver

2.5.1 Introduction

The two-dimensional world given in this chapter is useful for providing an understanding of what linear optimization is about, but it has very limited usefulness for practical problems. Real-world applications may involve thousands or even millions of decision variables. We won't be doing anything that big, but we do want to extend what we can do beyond just two variables. To do this requires an *algorithm*, which is a structured sequential approach for solving a problem. There are several algorithms for linear optimization, but the one most commonly used is called the *simplex algorithm*. At one time, learning the basics of how the simplex algorithm works was a core topic of the compulsory introductory course. Now, if taught at all, it would be in an elective course.

The simplex algorithm has been used in off-the-shelf software that has been written for optimization. This document concentrates on using the Solver within Excel for optimization. It should be noted, however, that most models of commercial size are run on software which is dedicated to linear optimization; for a discussion about such software see Appendix A which begins on page 459. One issue is that of speed when solving large models. Another reason is that dedicated software is generally better for data entry. However, in the educational context where we are only trying to solve small models, spreadsheets are quite handy. A big advantage is that if someone has already obtained an office suite for other purposes then there is nothing new to obtain. Also, a spreadsheet is particularly useful when the data are already in spreadsheet form.

2.5.2 Optimization using Spreadsheets

To use Excel for optimization there is an overall two-part process. At the outset, we must build a model in Excel. Then we invoke the Solver, which needs its own set of instructions.

Creating the Excel Model

The user begins by entering three types of information. First of all, there are labels. Secondly, there is the given numerical information of the problem. Thirdly, there are formulas. In what follows we enter the information in this order, but the information can be entered in any order.

This information is entered in a particular structure. In this book, the first column is used for labels and the computation of the OFV. After that, there is a column for each variable. Then comes a column for the computation of the left-hand side of each constraint. This is followed by labels for the direction of the inequality (or an equal sign for an equality constraint). Finally there is a column for the right-hand side values. To illustrate this, we will use the original formulation of the cement example:

$$\begin{aligned} X_1 &= \text{ the number of TPD of Type 1 cement made} \\ X_2 &= \text{ the number of TPD of Type 2 cement made} \end{aligned}$$

$$\begin{array}{rrcrcr} \text{maximize} & 8X_1 & + & 10X_2 & & \\ \text{subject to} & & & & & \\ \text{Type 1 Sales} & X_1 & & & \geq & 40 \\ \text{Type 2 Sales} & & & X_2 & \geq & 30 \\ \text{Total Production} & X_1 & + & X_2 & \leq & 200 \\ \text{Dept. A Labour} & 3X_1 & + & 2X_2 & \leq & 585 \\ \text{Dept. B Labour} & 1.5X_1 & + & 5X_2 & \leq & 500 \\ \text{Dept. C Labour} & 4X_1 & + & 6X_2 & \leq & 900 \\ \text{non-negativity} & X_1 & , & X_2 & \geq & 0 \end{array}$$

Since column A will be reserved for labels and the OFV, the two variables are represented by columns B and C, and the right-hand side values will appear in column F.

Labels are used to help make the model understood to the user and other persons who may look at the spreadsheet. Any cell containing a label has no effect on the calculations. Some of these labels are obvious, such as "Tonnes per Day" and "Total Production". However, there is also a column which gives the direction of the inequality of the constraints, be it <= to mean $\leq$, or >= to mean $\geq$, or = for an equality constraint.[9] These may appear to be commands, but they are simply labels. To the right of this is column F, which is headed by RHS (right-hand-side).

Entering all the labels we have:

[9] It is also possible to insert the special symbols $\leq$ and $\geq$. On the ribbon click on "Insert", and then on the extreme right click on "Symbol", search for the symbol and click on it, and then click on the "Insert" button.

	A	B	C	D	E	F
1		**Cement**	**Model**			
2	OFV	X1	X2			
3		Type 1	Type 2			
4	Maximize					
5	Tonnes per Day					
6						
7	Constraints					RHS
8	Type 1 Sales				>=	
9	Type 2 Sales				>=	
10	Total Production				<=	
11	Dept. A Labour				<=	
12	Dept. B Labour				<=	
13	Dept. C Labour				<=	

Now we enter the numerical information. The right-hand side values in column F are easy. The other numbers come from extracting the numbers from the objective function and the constraints. The objection function is to maximize $8X_1 + 10X_2$, hence the numbers are 8 and 10, which are placed in the columns for the X_1 and X_2 variables near the top of these columns. For the constraints, we must recognize that an X_1 is a $1X_1$, so its coefficient is 1. If a variable is missing from a row, then its coefficient is 0. With zeroes we have a choice: we can either enter a 0, or leave the cell blank. For this example we input the 0's, but for larger models it's simpler to leave such cells blank.

We must leave space in a row for the numerical values of the variables. Also, one cell is reserved for the value of the OFV. These values are not entered by the user; they will be calculated by the Solver. It is the practice in this document to highlight the reserved space for the variables in yellow, and the space for the OFV in green. Including the input data and the coloured cells the spreadsheet is now as follows:

	A	B	C	D	E	F
1		**Cement**	**Model**			
2	OFV	X1	X2			
3		Type 1	Type 2			
4	Maximize	8	10			
5	Tonnes per Day					
6						
7	Constraints					RHS
8	Type 1 Sales	1	0		>=	40
9	Type 2 Sales	0	1		>=	30
10	Total Production	1	1		<=	200
11	Dept. A Labour	3	2		<=	585
12	Dept. B Labour	1.5	5		<=	500
13	Dept. C Labour	4	6		<=	900

On a spreadsheet, the dot product of two rows is made using the **SUMPRODUCT** function.[10] The space reserved for the values of the variables is used by the objective function and by every constraint. In each constraint row, the value calculated by the **SUMPRODUCT** function goes to the right of the left-hand side data.

We calculate the OFV by using the **SUMPRODUCT** function, and we also use this function to calculate the numerical value of the left-hand side of each constraint. These numerical values must obey the relationship of the constraint. To save work, we can enter the **SUMPRODUCT** function for the OFV using absolute labels for the range containing the variables, and then copy the formula to where it is used by the constraints.

As the formulas are entered, we just see zeroes in those cells, because the yellow cells on which the calculations are based are all blank, and therefore the yellow cells are treated as zeroes. Once the Solver calculates numbers for the yellow cells, it will put the computed numbers in the formula cells. When we see these numbers we are in what is called *normal view*. However we might wish to see the formulas instead, and so we would switch to *formula view* (the procedure for switching is described on page 9).

[10]The syntax was seen in Chapter 1, and it can also be found using the *Help* menu. This function can also handle more than just a dot product. In Excel, a comma or asterisk is used to separate one range from another. It should also be noted that the **SUM** function could do this calculation, but would have to be defined as an array.

Here is the unsolved model, in normal view:

	A	B	C	D	E	F
1		**Cement**	**Model**			
2	OFV	X1	X2			
3	0	Type 1	Type 2			
4	Maximize	8	10			
5	Tonnes per Day					
6						
7	Constraints					RHS
8	Type 1 Sales	1	0	0	>=	40
9	Type 2 Sales	0	1	0	>=	30
10	Total Production	1	1	0	<=	200
11	Dept. A Labour	3	2	0	<=	585
12	Dept. B Labour	1.5	5	0	<=	500
13	Dept. C Labour	4	6	0	<=	900

The formula for cell A3 is `=SUMPRODUCT(B4:C4,B5:C5)`. Column A in formula view is:

	A
1	
2	OFV
3	=SUMPRODUCT(B4:C4,B5:C5)
4	Maximize
5	Tonnes per Day
6	
7	Constraints
8	Type 1 Sales
9	Type 2 Sales
10	Total Production
11	Dept. A Labour
12	Dept. B Labour
13	Dept. C Labour

The formula for cell D8 is `=SUMPRODUCT(B$5:C$5,B8:C8)`; this is copied into the cells below it in Column D. Here is column D in formula view:

	D
1	
2	
3	
4	
5	
6	
7	
8	=SUMPRODUCT($B$5:$C$5,B8:C8)
9	=SUMPRODUCT($B$5:$C$5,B9:C9)
10	=SUMPRODUCT($B$5:$C$5,B10:C10)
11	=SUMPRODUCT($B$5:$C$5,B11:C11)
12	=SUMPRODUCT($B$5:$C$5,B12:C12)
13	=SUMPRODUCT($B$5:$C$5,B13:C13)

In optimizing a model, we let Excel choose the values of the variables. To do this, we need to use the spreadsheet Solver. The overview provided here should be sufficient, but if needed a Solver tutorial is available from Frontline Systems, Inc. at https://www.solver.com/.

Installing the Solver

On Windows If the Solver has not already been installed, the installation in Windows is accessed as follows:

1. Click the “File” tab (top left of the screen).

2. A menu will appear of the left of the screen. Click on the “Options” tab near the bottom.

3. A screen will appear called “Excel Options”. On the left, near the bottom, click on “Add-Ins”.

4. In the main body of the screen, there will be the word “Manage:”. Set the box to its right to “Excel Add-Ins”, and then click on the “GO” button to the right.

5. An “Add-Ins” screen will appear. Click on the box to the left of the words “Solver Add-In”, and then click on “OK”.

On Apple Mac For the Apple Mac the procedure is:

1. Choose “Tool” on the TOP menu.

2. Select “Excel Add-ins..” from the menu.

3. Select “Solver Add-In” on the panel.

4. Solver is accessed from the “Data” MAIN menu.

Using the Solver

After entering the model, the Solver is invoked by clicking on **Data**, and then on the far right, clicking on **Solver**. The user specifies the following:

1. the cell which is to be optimized, called the *objective cell*, which is the cell which will contain the OFV)

2. the objective (e.g. maximization)

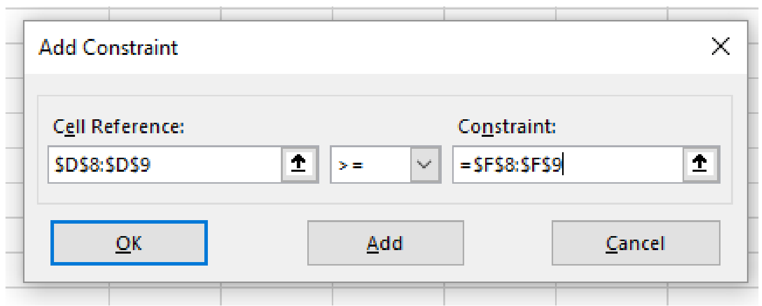
3. a range of cells which the Solver may vary, i.e. the range of cells reserved for the values of the variables, called the *variable cells*, and

4. the constraints.

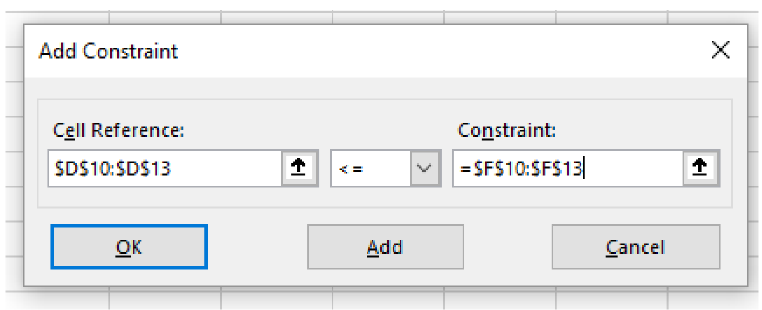
For every constraint we will compare the cell which contains the value of the left hand side with the cell which contains the right hand side value, specifying the relationship ($\leq$, =, or $\geq$) between these two cells. Constraints which are contiguous to one another of the same type ($\leq$, =, or $\geq$) can be entered as a range rather than specifying each one separately.

In the Solver window, the user needs to click on the box next to the words “Make Unconstrained Variables Non-Negative”. Also, after the words “Select a Solving Method”, the user should choose “Simplex LP”. In the Options box, the default numerical values should be fine, and the three boxes should be blank, except that the first one "Use Automatic Scaling" may be desired.[11]

For this example, the objective cell is A3, the objective is maximization, and the variable cells are in the range B5:C5. We click on the "Add" button to add a set of constraints. Since the first two constraints are both $\geq$, we add both at the same time, clicking and then dragging the mouse over the two cells on both the left-hand and right-hand sides. We require that the number in cell D8, which is the numerical value of the left-hand side of the first constraint, be $\geq$ the number in cell F8, and also that the number in cell D9 be $\geq$ the number in cell F9. In the middle, we need to set the direction to be ">=" (the default is "<=").

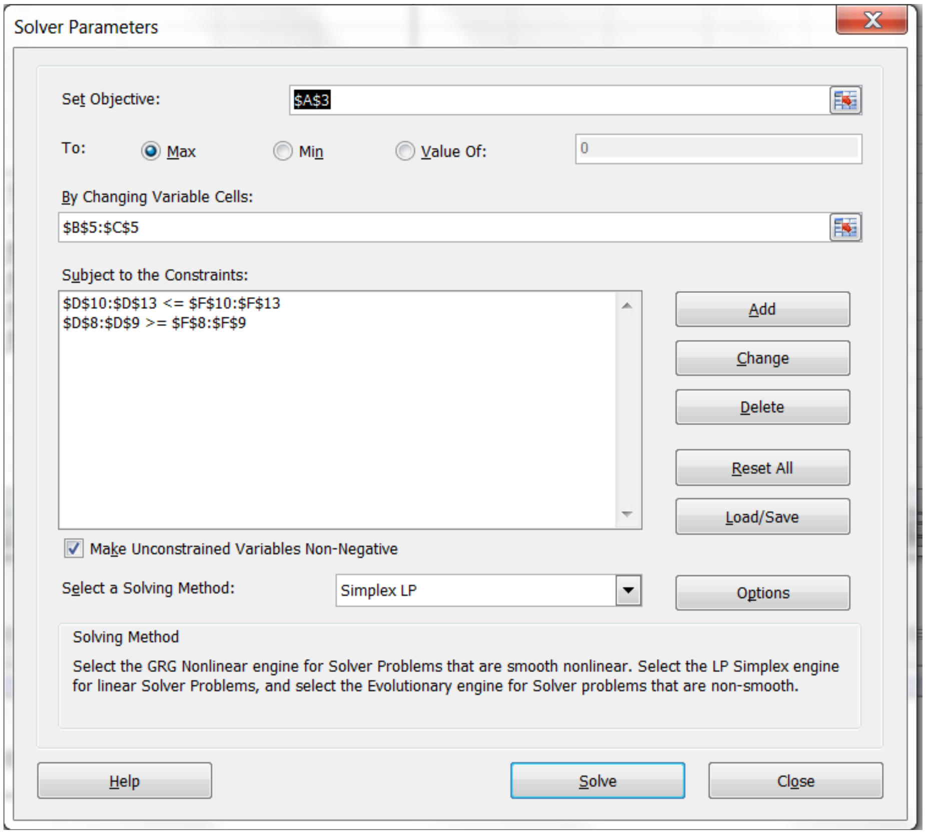


The last four constraints are all $\leq$, so they are entered as:



Filling in the Solver we have:

[11] A model is said to be *poorly scaled* when the coefficients of one row are very much greater than those of another, for example if one constraint is $2X_1 + 5X_2 \leq 41$ while another is $450,000X_1 + 195,000X_2 \leq 2,715,000$. When the Solver tries to solve a poorly scaled model, it may experience numerical problems in finding the optimal solution. Automatic rescaling helps eliminate such problems. See D. Flystra, A. Lasdon, J. Watson, and A. Waren, "Design and Use of the Microsoft Excel Solver", *Interfaces*, **28**:5 September-October 1998, pp. 29-55.



Optimizing the model we obtain (the optimal values of the variables are highlighted):

	A	B	C	D	E	F
1		**Cement**	**Model**			
2	OFV	X1	X2			
3	1700	Type 1	Type 2			
4	Maximize	8	10			
5	Tonnes per Day	150	50			
6						
7	Constraints					RHS
8	Type 1 Sales	1	0	150	>=	40
9	Type 2 Sales	0	1	50	>=	30
10	Total Production	1	1	200	<=	200
11	Dept. A Labour	3	2	550	<=	585
12	Dept. B Labour	1.5	5	475	<=	500
13	Dept. C Labour	4	6	900	<=	900

As one would expect, cell A3 contains the optimal OFV of 1700, and cells B5 and C5 contain 150 and 50 respectively, which are the optimal Tonnes per Day of type 1 and type 2 cement respectively.

The user can request an "Answer Report" which will give the value of the target cell (the OFV), the values of all the variables, and the slack, if any, on each constraint. Note that in the Answer Report, the order of the constraints is not the same as the order in the algebraic model.

Omitting what appears at the top of the Answer Report we have:

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$A$3	OFV	0	1700

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$5	Tonnes per Day Type 1	0	150	Contin
$C$5	Tonnes per Day Type 2	0	50	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$10	Total Production	200	$D$10<=$F$10	Binding	0
$D$11	Dept. A Labour	550	$D$11<=$F$11	Not Binding	35
$D$12	Dept. B Labour	475	$D$12<=$F$12	Not Binding	25
$D$13	Dept. C Labour	900	$D$13<=$F$13	Binding	0
$D$8	Type 1 Sales	150	$D$8>=$F$8	Not Binding	110
$D$9	Type 2 Sales	50	$D$9>=$F$9	Not Binding	20

2.5.3 Slack and Surplus

The Excel output contains a column labelled “Slack”, which needs some explanation.

We begin by defining two concepts, that of *slack* and *surplus*, as they appear in most books and in most software for optimization (but not on the Excel Solver). For a $\leq$ constraint, the slack is defined as the right-hand side value minus the value of the left-hand side at the point of optimality. For a $\geq$ constraint, the surplus is defined as the value of the left-hand side at the point of optimality minus the right-hand side value. By contrast, the Excel Solver only uses the term slack; for either a $\leq$ or a $\geq$ constraint, the Solver computes the slack for each constraint as the absolute value of the difference between the right-hand side value and the numerical value of the left-hand-side computed at the optimal solution.

On page 50, we saw that a *binding* constraint is one that passes through the optimal solution. An equivalent definition is that if the slack or surplus is 0, then the constraint is said to be *binding*; if the slack or surplus is greater than 0, then the constraint is non-binding. Said another way, a constraint is binding if and only if the value of the left-hand side at the point of optimality equals the right-hand side value.

For a model for which the optimal solution has been computed, the slack or surplus can easily be found by hand. For the cement example we know that the optimal solution is $X_1 = 150$ and $X_2 = 50$. Let's calculate the surplus on the Type 1 Sales constraint, and the slack on the Department A Labour constraint. The Type 1 Sales constraint is $X_1 \geq 40$ (or $1X_1 + 0X_2 \geq 40$). The optimal values of X_1 and X_2 are 150 and 50 respectively, hence the value of the left-hand side of the constraint is 1(150) + 0(50) = 150. Since the number on the right-hand side is only 40, the left-hand side value is $150 - 40 = 110$ more than it needs to be; the surplus on the Type 1 Sales Constraint is 110. The Department A Labour constraint is $3X_1 + 2X_2 \leq 585$. At the optimal solution of (150,50) the left-hand side value is:

$$3(150) + 2(50) = 550$$

By subtracting 550 from 585, we obtain a slack of 35.

2.6 Model Variations with Three Variables (Optional)

2.6.1 Cement Problem

Here we consider the cement model with the proportion constraint, using three variables. Two of them were defined earlier; the third one is:

$$X_3 \quad = \quad \text{the total production of cement in TPD}$$

The variable X_1 cannot exceed two-thirds of X_3, which we write as

$$X_1 \leq \tfrac{2}{3} X_3$$

or equivalently

$$X_1 - \tfrac{2}{3} X_3 \leq 0$$

We need to add this *proportion* constraint to the existing model. Also, we need to write the relationship between X_3 and the other variables, which is:

$$X_3 = X_1 + X_2$$

As a constraint with all variables on the left, this is:

$$X_3 - X_1 - X_2 = 0$$

which can be re-arranged to:

$$X_1 + X_2 - X_3 = 0$$

With this third variable present the total production constraint can be written in terms of it. Doing this and then adding the new third variable and the two new constraints we obtain:

$X_1 =$ the number of TPD of Type 1 cement made
$X_2 =$ the number of TPD of Type 2 cement made
$X_3 =$ the total production of cement in TPD

$$\begin{array}{rrcrcrcr}
\text{maximize} & 8X_1 & + & 10X_2 & & & & \\
\text{subject to} & & & & & & & \\
\text{Type 1 Sales} & X_1 & & & & & \geq & 40 \\
\text{Type 2 Sales} & & & X_2 & & & \geq & 30 \\
\text{Total Production} & & & & & X_3 & \leq & 200 \\
\text{Dept. A Labour} & 3X_1 & + & 2X_2 & & & \leq & 585 \\
\text{Dept. B Labour} & 1.5X_1 & + & 5X_2 & & & \leq & 500 \\
\text{Dept. C Labour} & 4X_1 & + & 6X_2 & & & \leq & 900 \\
\text{Proportion} & X_1 & & & - & \frac{2}{3}X_3 & \leq & 0 \\
\text{Balance} & X_1 & + & X_2 & - & X_3 & = & 0 \\
\text{non-negativity} & X_1 & , & X_2 & , & X_3 & \geq & 0
\end{array}$$

Modeling in this manner is the best form in that no calculations are required for any of the parameters. The advantages of doing no calculations are twofold: the original data are preserved; and there's less likely to be a mistake. To solve this model using the Excel Solver, we do not even need to make the minor calculation of converting the minus two-thirds to decimal form, i.e. -0.66667; all we need to do is enter `=-2/3` into the appropriate cell.

2.6.2 Diet Model

An alternate way of handling the iron proportion constraint is to define a third variable, used to represent the total amount of iron consumed:

$X_3 =$ the amount of iron consumed each day (in mg)

The daily intake of iron from hamburgers (in mg) is $5.547X_1$. Hence we must have:

$$\begin{aligned} 5.547X_1 &\leq 0.9X_3 \\ 5.547X_1 - 0.9X_3 &\leq 0 \end{aligned}$$

The total iron intake X_3 is the amount from hamburgers, which is $5.547X_1$, plus the amount from orange juice, which is $1.096X_2$. Therefore, we must have:

$$X_3 = 5.547X_1 + 1.096X_2$$

which we can re-arrange as

$$5.547X_1 + 1.096X_2 - X_3 = 0$$

Finally, we have the non-negativity restrictions.

Summary

The completed model is:

$X_1 =$ the number of double hamburgers eaten each day
$X_2 =$ the number of servings of orange juice drunk each day
$X_3 =$ the amount of iron consumed each day (in mg)

$$\begin{array}{rrcrcrcr} \text{minimize} & 1.25X_1 & + & 0.32X_2 & & & & \\ \text{subject to} & & & & & & & \\ \text{Protein RDI} & 31.820X_1 & + & 1.469X_2 & & & \geq & 46 \\ \text{Iron RDI} & 5.547X_1 & + & 1.096X_2 & & & \geq & 15 \\ \text{Vitamin C RDI} & 1.075X_1 & + & 85.656X_2 & & & \geq & 60 \\ \text{Iron Proportion} & 5.547X_1 & & & - & 0.9X_3 & \leq & 0 \\ \text{Iron Balance} & 5.547X_1 & + & 1.096X_2 & - & X_3 & = & 0 \\ \text{non-negativity} & X_1 & , & X_2 & , & X_3 & \geq & 0 \end{array}$$

Because of the third variable, we would need to solve this model using the Solver on Excel.[12]

[12]Normally, a model with three variables cannot be converted to a model with just two variables. However, when there is an equality constraint, one variable can be written in terms of the other two variables, and then this one variable can be eliminated. To do so would produce the models which we solved by graphing earlier in this chapter.

2.7 Problems for Student Completion

Using just two variables, formulate a linear optimization model for each of the following problems. Solve each model graphically, clearly indicating the feasible region, and both the trial and optimal isovalue lines. For each model, use algebra to determine the exact solution for the variables and the objective function value. You may find it useful to also solve one or more of these problems on a spreadsheet.

2.7.1 Garment Problem

When solving the following model, use a piece of graph paper with each axis labelled from 0 to 300, and draw all lines within the 300 by 300 grid.

A garment factory makes blouses and dresses. Each blouse gives a profit of \$2, while each dress gives a profit of \$3. They can sell at most 190 dresses. Each garment spends time on three machines as follows:

	Minutes per Garment		Minutes
Machine	Blouse	Dress	Available
Cutting	3	6	1413
Sewing	6	2	1218
Assembly	5	4	1317

The number of dresses must be at least 30% of the total number of garments made.

2.7.2 Baseball Bat Problem

A baseball bat company makes two models, the "slugger" and the "whacker". Each slugger requires four minutes of lathework, and one minute of varnishing. Each whacker requires five minutes of lathework and 45 seconds of varnishing. Each day, the combined production cannot exceed 980 bats. The woodworking shop operates 16 hours/day, with one room containing five lathes, and one varnishing room. Each of the five lathes is available for 55 minutes each hour, and the varnishing room is available for 50 minutes each hour. Each slugger contributes \$5 to profit, and each whacker contributes \$6.

The company wishes to determine how many sluggers and whackers should be made each day.

2.7.3 Car-Assembly

A car-assembly plant makes sedans and SUVs. Each sedan gives a profit of \$500, while each SUV gives a profit of \$800. They can sell at most 550 sedans. Each vehicle spends time on three operations as follows:

	Hours per Vehicle		Hours
Operation	Sedan	SUV	Available
Assembly	12	24	16,956
Welding	6	2	3,654
Painting	5	4	3,951

SUVs must comprise at least 40% of the total number of vehicles made.

(a) Using just two variables, formulate a linear optimization model for this problem.

(b) Without drawing any lines outside a 1000 by 1000 grid, solve the model from part (a) graphically, clearly indicating the feasible region, and both the trial and optimal isovalue lines. Use algebra to determine the exact solution, and state the recommendation.

2.7.4 Quarry Problem

Background Note: The *density* of an object is its mass divided by its volume. Hence mass is density times volume, and volume is mass divided by density.

Two types of rock are mined in a quarry. “Softrock” has an density of 5 Tonnes per cubic metre, and “hardrock” has a density of 8 Tonnes per cubic metre. Up to 600 Tonnes of softrock can be mined each hour, and independent of this, up to 300 Tonnes of hardrock can be mined each hour. The mined rock is crushed and then travels on a conveyor belt. (To avoid mixing the two types of rock they will crush one type of rock, then switch over to the other type, and then keep switching back and forth).

The conveyor belt can handle up to 110 cubic metres of rock per hour. The crusher can handle up to 1000 Tonnes per hour when crushing softrock, or up to 400 Tonnes per hour when crushing hardrock. The company makes \$10/Tonne for softrock and \$14/Tonne for hardrock. The quarry operator wishes to know how many Tonnes of each type of rock they should produce each hour.

2.7.5 Office Rental

A company needs to rent space for its office employees both in the suburbs and downtown. Space is available in suburbia at a rate of \$100 per square metre (per annum), while downtown space rents for \$210 per square metre (per annum). In suburbia, only 30% of the space is "executive" quality, while the rest is ordinary quality. At the downtown location, 60% of the space is executive quality, while the rest is ordinary quality. The company needs a total of at least 900 square metres of space, of which at least 420 square metres must be executive quality. No more than three quarters of the entire space is to be at either location.

They wish to know how much space they should rent in each place so as to minimize the total expenditure on rent. Formulate and solve by the graphical method.

2.7.6 Diet Problem

Suppose that a kilogram of beef contains 600 grams of protein, and 80 grams of fat, but no Vitamin C. A litre of orange juice contains 6 grams of protein, no fat, and four times the required daily intake of Vitamin C. A person needs 54 grams of protein per day, and the fat should be between 10 and 60 grams per day. No more than 95% of the protein consumed should come from beef. A kilogram of beef costs \$6, while a litre of orange juice costs \$2.

Based on these two foods alone, and only the stated requirements, we seek the minimum cost daily diet. Formulate and solve by the graphical method.

Chapter 3

Applications of Linear Models

The formulation examples of the previous chapter were relatively easy, mostly because the decision variables were obvious. There are many applications of linear optimization where this is not the case. In this chapter we shall examine several applications where we must give a fair deal of thought as to what the decision variables should be.

An algebraic model is created for each problem. For some of these problems, we also provide the corresponding spreadsheet models.

3.1 Blending

Here we give an example of a common use of linear optimization from the oil and gas industry – that of blending several inputs to produce several outputs. First of all, we will discuss a bit of chemistry.

3.1.1 Background Information

There are many characteristics of gasoline which one could measure. For our purposes, we will just use two, *octane rating* and *vapour pressure*.

The higher the octane rating of a gasoline, the greater the anti-knock properties. The rating is expressed without units, with most gasolines sold commercially having an octane rating between 80 and 110. Higher octane gasoline is required for high performance engines found in some cars and in the aviation industry.

The vapour pressure is the pressure exerted by the gasoline's vapour on the liquid gasoline. Higher performance engines usually require a *lower* vapour pressure than ordinary engines. It is usually measured in kilopascals (kPa). [1]

When two gasolines are blended, the octane rating of the blend is a function of the octane ratings of each gasoline. As an approximation, [2] it can be taken as the weighted average, where the weights are the volumes of the inputs. For example, if 7 litres of an 83 octane gasoline are mixed with 13 litres of a 101 octane gasoline, then the octane rating of the 20 litres of mixed gasoline is about

$$\frac{7 \times 83 + 13 \times 101}{7 + 13} = 94.7$$

The vapour pressure of the blend can be found in a similar manner.

3.1.2 Problem Description

Returning to linear optimization, we now consider a situation where two gasolines are blended into two commercial products. We will refer to these as input gasolines 1 and 2 and output gasolines 1 and 2 respectively. For the inputs, the octane ratings, the vapour pressures in kilopascals, and the amounts available in cubic metres and their prices are known. These are:

Input Gasoline #	Octane Rating	Vapour Pressure (kPa)	Amount Available (m^3)	Price (\$ per m^3)
1	110	35	25,000	265
2	80	65	60,000	188

For the output gasolines, the company has made a set of specifications. There is of course no need to produce products at the limit of the specifications. If, for example, a minimum octane rating of 95 is promised, there is nothing wrong with delivering it to the customer with a rating of 96.3. We further suppose that the company must make a minimum amount of each type of output to serve its customer base. These data, and the wholesale (before tax) prices are:

[1] A kilopascal is the pressure exerted by a force of one thousand newtons over an area of one square metre.

[2] Chemistry is an empirical science – sometimes our intuition does not help us. For example, the freezing point of a equal-part mixture of water and ethylene glycol is lower than the freezing point of either water or ethylene glycol.

Output Gasoline #	Minimum Octane Rating	Maximum Vapour Pressure (kPa)	Minimum Amount Required (m^3)	Wholesale Price (\$ per m^3)
1	95	40	15,000	310
2	85	55	30,000	230

3.1.3 Formulation

With Subscripted Variables

We begin by creating a model using subscripted variables, For large problems this is the only practical way to make the algebraic model. That being said, an example as small as this one can be formulated with unsubscripted variables, and this variant is presented in the next section.

As always, we begin by trying to define the decision variables. We say "try" because unless one has seen a problem like this before, or unless one is particularly clever, it is difficult to define all the variables at the outset. What we will do is define the ones which are obvious, then attempt to write the constraints, and by doing this seeing what other variables we need.

The place to start is the objective. We are trying to maximize the contribution to profit from the blending of the gasolines. This contribution is the revenue from the sales of the two output gasolines, minus the cost of the purchases of the two input gasolines. Therefore, we wish to know the volume produced and sold of each output gasoline, and we wish to know the volume purchased of each input gasoline. At a minimum, we need the following four decision variables:

$X_1 =$ amount (in m^3) of output gasoline #1 sold,
$X_2 =$ amount (in m^3) of output gasoline #2 sold,
$I_1 =$ amount (in m^3) of input gasoline #1 purchased,
$I_2 =$ amount (in m^3) of input gasoline #2 purchased.

We can now write the objective function. To determine the profit, the cost of the input gasolines must be subtracted from the revenue of the output gasolines. Hence, the coefficients of the two input gasolines will be negative.

$$\text{maximize } 310X_1 + 230X_2 - 265I_1 - 188I_2$$

Some of the constraints can now be written. These are the constraints on the availability of the inputs, and the minimum production constraints on the outputs.

$$\begin{array}{rcl} I_1 & \leq & 25,000 \\ I_2 & \leq & 60,000 \\ X_1 & \geq & 15,000 \\ X_2 & \geq & 30,000 \end{array}$$

Unfortunately, this is as far as we can go with these four variables. In order to write the constraints on the octane rating and the vapour pressure of the two output gasolines, we will have to define some more variables. In order to compute the octane rating of one of the outputs, we need to know the amount which comes from each input. The same applies for the octane rating of the other output, and the vapour pressure of each of the two outputs. We therefore need to know how much of input 1 is used to make output 1, how much of input 1 is used to make output 2, how much of input 2 is used to make output 1, and how much of input 2 is used to make output 2. Hence, we need another four variables. Rather than define these variables as U_1 to U_4, we will use double subscription. The first index refers to the input, and the second index refers to the output. Hence:

$U_{1,1}$ = amount (in m^3) of input 1 used to make output 1,

$U_{1,2}$ = amount (in m^3) of input 1 used to make output 2,

$U_{2,1}$ = amount (in m^3) of input 2 used to make output 1,

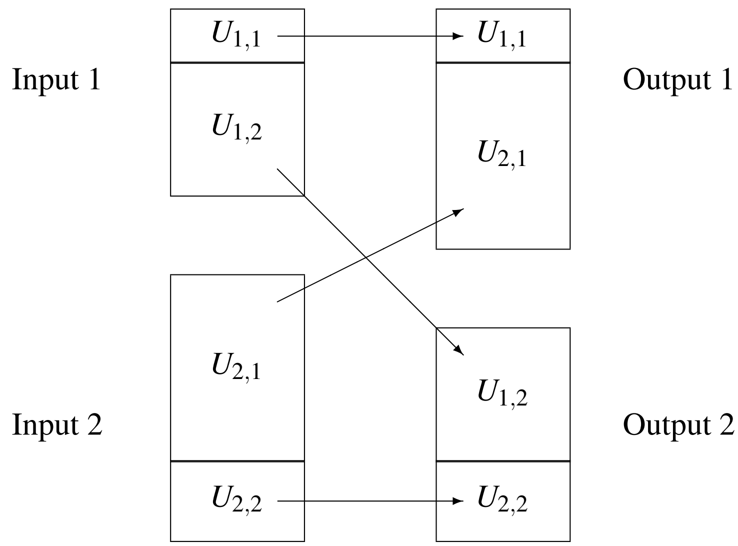
$U_{2,2}$ = amount (in m^3) of input 2 used to make output 2.

A compact way of writing these four definitions is to define

U_{ij} = amount (in m^3) of input i used to make output j, where $i = 1,2$ and $j = 1,2$.[3]

The following picture illustrates how the U_{ij}'s are transferred from the inputs to the outputs.

[3] With double subscription, we need to use a comma separator when use numbers. However, when using i and j, we do not need a comma separator. Hence, we write U_{ij} rather than $U_{i,j}$.



The U_{ij}'s are related to the I_i's and the X_j's by what are known as *volume balance* constraints. For example, we must have that

$$U_{1,1} + U_{1,2} = I_1.$$

Keeping with the convention that all variables must appear on the left we will re-write this as

$$-I_1 + U_{1,1} + U_{1,2} = 0.$$

Addition is commutative, so we could have written $U_{1,1} + U_{1,2} - I_1 = 0$. Either way is acceptable as far as the formulation of the algebraic model is concerned. However, the order does matter when the model is converted into spreadsheet form. The order is established first by the objective function, which contains the variables X_1, X_2, I_1 and I_2. For the variables which do not appear in the objective function, the order is that of first appearance in the constraints.

The other balance constraints are:

$$\begin{aligned} -I_2 + U_{2,1} + U_{2,2} &= 0 \\ -X_1 + U_{1,1} + U_{2,1} &= 0 \\ -X_2 + U_{1,2} + U_{2,2} &= 0 \end{aligned}$$

The octane rating of output 1 is expressed[4] as:

$$\frac{110U_{1,1}+80U_{2,1}}{U_{1,1}+U_{2,1}}$$

Since the octane rating of output gasoline 1 must be at least 95, we write:

$$\frac{110U_{1,1}+80U_{2,1}}{U_{1,1}+U_{2,1}} \geq 95$$

We could simplify this non-linear expression by multiplying both sides of the inequality by $U_{1,1}+U_{2,1}$, and then subtracting $95(U_{1,1}+U_{2,1})$ from both sides to obtain:

$$15U_{1,1}-15U_{2.1} \geq 0$$

However, doing things this way hides the original data of the problem. Instead, we make the substitution of X_1 for $U_{1,1}+U_{2,1}$:

$$\frac{110U_{1,1}+80U_{2,1}}{X_1} \geq 95$$

Now we multiply both sides by X_1 to obtain:

$$110U_{1,1}+80U_{2,1} \geq 95X_1.$$

Not only is this now in a linear form, this expression also permits the possibility of X_1 being 0. Putting all variables on the left we obtain:

$$-95X_1+110U_{1,1}+80U_{2,1} \geq 0.$$

We could have written $110U_{1,1}+80U_{2,1}-95X_1 \geq 0$, but as we did for the equality constraints, we use the variable order which will be required when the model is converted into spreadsheet form.

Similarly, the octane rating constraint for output 2 gasoline is:

$$\frac{110U_{1,2}+80U_{2,2}}{U_{1,2}+U_{2,2}} \geq 85$$

Substituting X_2 for $U_{1,2}+U_{2,2}$, cross-multiplying, and then subtracting, we obtain:

$$-85X_2+110U_{1,2}+80U_{2,2} \geq 0.$$

[4] Assuming that the denominator is greater than 0.

The vapour pressure of output 1 gasoline must be no more than 40 kPa:

$$\frac{35U_{1,1} + 65U_{2,1}}{U_{1,1} + U_{2,1}} \leq 40$$

Substituting X_1 and then cross-multiplying gives:

$$35U_{1,1} + 65U_{2,1} \leq 40X_1$$

Subtracting the $40X_1$ from both sides gives:

$$-40X_1 + 35U_{1,1} + 65U_{2,1} \leq 0$$

The vapour pressure for output 2 gasoline can be no more than 55 kPa:

$$\frac{35U_{1,2} + 65U_{2,2}}{U_{1,2} + U_{2,2}} \leq 55$$

After substituting X_2 and re-arranging the vapour pressure constraint for output 2 gasoline is:

$$-55X_2 + 35U_{1,2} + 65U_{2,2} \leq 0$$

Finally, we require that all variables be non-negative.[5]

Putting all the above together we obtain:

X_1 = amount (in m^3) of output gasoline #1 sold,
X_2 = amount (in m^3) of output gasoline #2 sold,
I_1 = amount (in m^3) of input gasoline #1 purchased,
I_2 = amount (in m^3) of input gasoline #2 purchased,
$U_{1,1}$ = amount (in m^3) of input 1 used to make output 1,
$U_{1,2}$ = amount (in m^3) of input 1 used to make output 2,
$U_{2,1}$ = amount (in m^3) of input 2 used to make output 1,

[5] The non-negativity restrictions for X_1 and X_2 are technically redundant, but we include them anyway. If, for example, the minimum sales constraints were removed, the non-negativity restrictions would have to be there.

$U_{2,2}$ = amount (in m^3) of input 2 used to make output 2.

$$
\begin{array}{rrcl}
\text{maximize} & 310X_1 + 230X_2 - 265I_1 - 188I_2 & & \\
\text{subject to} & & & \\
\text{Available, Input 1} & I_1 & \leq & 25000 \\
\text{Available, Input 2} & I_2 & \leq & 60000 \\
\text{Minimum production, Output 1} & X_1 & \geq & 15000 \\
\text{Minimum production, Output 2} & X_2 & \geq & 30000 \\
\text{Balance, Input 1} & -I_1 + U_{1,1} + U_{1,2} & = & 0 \\
\text{Balance, Input 2} & -I_2 + U_{2,1} + U_{2,2} & = & 0 \\
\text{Balance, Output 1} & -X_1 + U_{1,1} + U_{2,1} & = & 0 \\
\text{Balance, Output 2} & -X_2 + U_{1,2} + U_{2,2} & = & 0 \\
\text{Octane Rating, Output 1} & -95X_1 + 110U_{1,1} + 80U_{2,1} & \geq & 0 \\
\text{Octane Rating, Output 2} & -85X_2 + 110U_{1,2} + 80U_{2,2} & \geq & 0 \\
\text{Vapour Pressure, Output 1} & -40X_1 + 35U_{1,1} + 65U_{2,1} & \leq & 0 \\
\text{Vapour Pressure, Output 2} & -55X_2 + 35U_{1,2} + 65U_{2,2} & \leq & 0 \\
 & & & \\
 & \text{all variables must be} & \geq & 0
\end{array}
$$

Finally, we note that it is possible, but not recommended, to formulate this problem in a more compact fashion, by using the equality constraints to replace the X and I variables with the U variables. This would save us four constraints and four variables, but it makes the formulation more difficult and less intuitive. Therefore, we strongly prefer the form given here.

Alternate Notation

As mentioned earlier, an example as small as this one can be made with unsubscripted variables. Here follows an alternate way to name the variables:

X = amount (in m^3) of output gasoline #1 sold,
Y = amount (in m^3) of output gasoline #2 sold,
A = amount (in m^3) of input gasoline #1 purchased,
B = amount (in m^3) of input gasoline #2 purchased,
AX = amount (in m^3) of input A used to make output X,
AY = amount (in m^3) of input A used to make output Y,
BX = amount (in m^3) of input B used to make output X,
BY = amount (in m^3) of input B used to make output Y.

Do not confuse AX with A times X; AX is a single entity.

Using this notation the model is:

$$
\begin{array}{rrcl}
\text{maximize} & 310X + 230Y - 265A - 188B & & \\
\text{subject to} & & & \\
\text{Available, Input A} & A & \le & 25000 \\
\text{Available, Input B} & B & \le & 60000 \\
\text{Minimum production, Output X} & X & \ge & 15000 \\
\text{Minimum production, Output Y} & Y & \ge & 30000 \\
\text{Balance, Input A} & -A + AX + AY & = & 0 \\
\text{Balance, Input B} & -B + BX + BY & = & 0 \\
\text{Balance, Output X} & -X + AX + BX & = & 0 \\
\text{Balance, Output Y} & -Y + AY + BY & = & 0 \\
\text{Octane Rating, Output X} & -95X + 110AX + 80BX & \ge & 0 \\
\text{Octane Rating, Output Y} & -85Y + 110AY + 80BY & \ge & 0 \\
\text{Vapour Pressure, Output X} & -40X + 35AX + 65BX & \le & 0 \\
\text{Vapour Pressure, Output Y} & -55Y + 35AY + 65BY & \le & 0 \\
 & & & \\
 & \text{all variables must be} & \ge & 0
\end{array}
$$

3.1.4 Solution Using the Excel Solver

We put this formulation onto a spreadsheet. Here are a few points relevant to this example:

1. The variable names in the spreadsheet model use the original subscripted ones. However, these are just labels which mean nothing to Excel. We could easily use the alternate notation if we wish.

2. Since the SUMPRODUCT function works on a sum of products, we handle subtraction rather than addition by treating the coefficient as being negative. For example, we treat -265 as if it were $+(-265)$. The + sign is assumed in the SUMPRODUCT function, and the negative simply appears in the cell which contains the 265 (which is D3). The formula placed in cell A3 is `=SUMPRODUCT(B3:I3,B4:I4)`.

3. In Column J, we enter `=SUMPRODUCT($B$4:$I$4,B6:I6)` into cell J6, and then copy this into the range J6:J17.

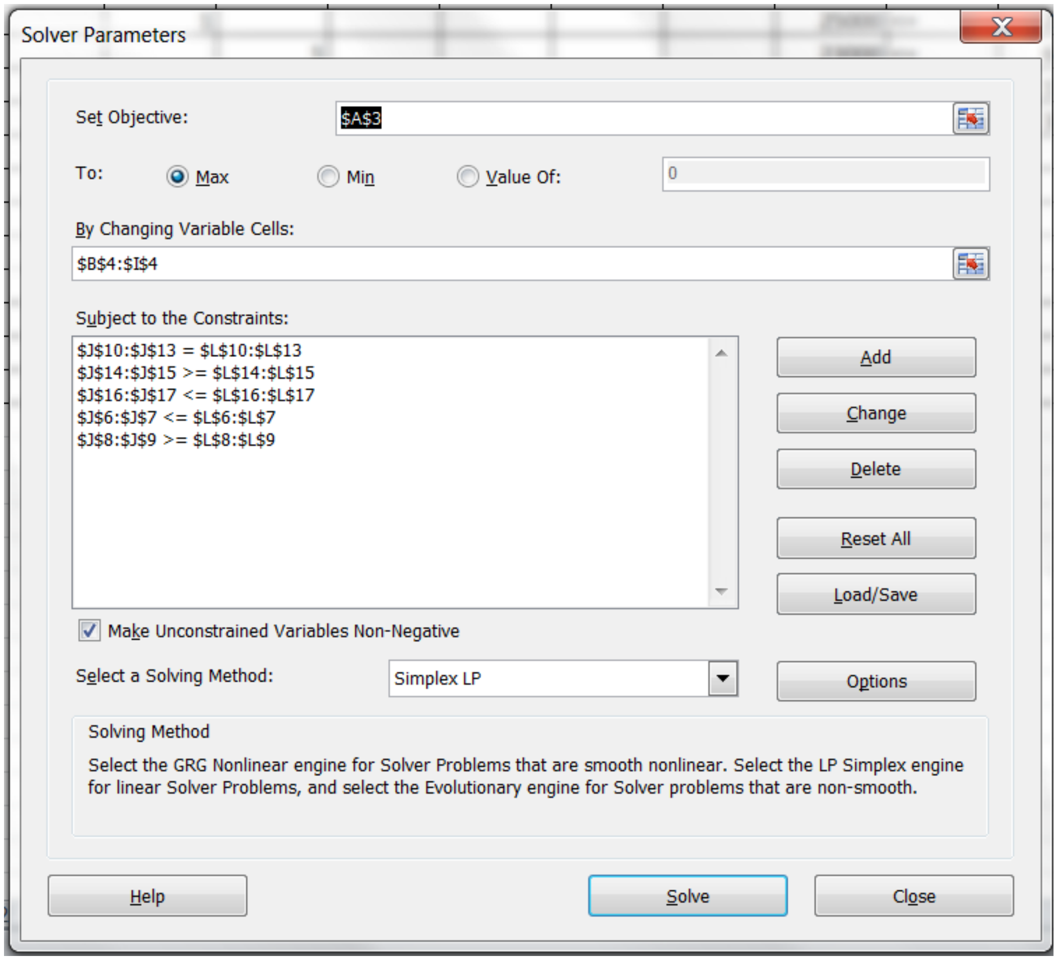
4. Excel assumes that an "=" sign begins an equation; when it doesn't (as used in K10:K13), we need to enter an apostrophe before the equal sign (i.e. '= creates a visual = sign in Excel).

We enter the data to obtain:

	A	B	C	D	E
1	Blending Model	X1	X2	I1	I2
2	Profit	Output 1	Output 2	Input 1	Input 2
3	$0	310	230	-265	-188
4					
5					
6	Available, Input 1			1	
7	Available, Input 2				1
8	Minimum Prod., Output 1	1			
9	Minimum Prod., Output 2		1		
10	Balance, Input 1			-1	
11	Balance, Input 2				-1
12	Balance, Output 1	-1			
13	Balance, Output 2		-1		
14	Octane Rating, Output 1	-95			
15	Octane Rating, Output 2		-85		
16	Vapour Pressure, Output 1	-40			
17	Vapour Pressure, Output 2		-55		

	F	G	H	I	J	K	L
1	U11	U12	U21	U22			
2	1 into 1	1 into 2	2 into 1	2 into 2			
3	0	0	0	0			
4							RHS
5							
6					0	<=	25000
7					0	<=	60000
8					0	>=	15000
9					0	>=	30000
10	1	1			0	=	0
11			1	1	0	=	0
12	1		1		0	=	0
13		1		1	0	=	0
14	110		80		0	>=	0
15		110		80	0	>=	0
16	35		65		0	<=	0
17		35		65	0	<=	0

Entering consecutive constraints of the same type ($\leq$, $=$, or $\geq$) together, we obtain the following Solver Parameters box:



Using the Solver we obtain:

	A	B	C	D	E
1	Blending Model	X1	X2	I1	I2
2	Profit	Output 1	Output 2	Input 1	Input 2
3	$1,531,000	310	230	-265	-188
4		18000	30000	25000	23000
5					
6	Available, Input 1			1	
7	Available, Input 2				1
8	Minimum Prod., Output 1	1			
9	Minimum Prod., Output 2		1		
10	Balance, Input 1			-1	
11	Balance, Input 2				-1
12	Balance, Output 1	-1			
13	Balance, Output 2		-1		
14	Octane Rating, Output 1	-95			
15	Octane Rating, Output 2		-85		
16	Vapour Pressure, Output 1	-40			
17	Vapour Pressure, Output 2		-55		

	F	G	H	I	J	K	L
1	U11	U12	U21	U22			
2	1 into 1	1 into 2	2 into 1	2 into 2			
3	0	0	0	0			
4	15000	10000	3000	20000			RHS
5							
6					25000	<=	25000
7					23000	<=	60000
8					18000	>=	15000
9					30000	>=	30000
10	1	1			0	=	0
11			1	1	0	=	0
12	1		1		0	=	0
13		1		1	0	=	0
14	110		80		180000	>=	0
15		110		80	150000	>=	0
16	35		65		0	<=	0
17		35		65	0	<=	0

We can simply read the solution from the output, or we can ask for the Answer Report, which is:

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$A$3	Profit	$0	$1,531,000

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$4	Output 1	0	18000	Contin
$C$4	Output 2	0	30000	Contin
$D$4	Input 1	0	25000	Contin
$E$4	Input 2	0	23000	Contin
$F$4	1 into 1	0	15000	Contin
$G$4	1 into 2	0	10000	Contin
$H$4	2 into 1	0	3000	Contin
$I$4	2 into 2	0	20000	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$J$10	Balance, Input 1	0	$J$10=$L$10	Binding	0
$J$11	Balance, Input 2	0	$J$11=$L$11	Binding	0
$J$12	Balance, Output 1	0	$J$12=$L$12	Binding	0
$J$13	Balance, Output 2	0	$J$13=$L$13	Binding	0
$J$14	Octane Rating, Output 1	180000	$J$14>=$L$14	Not Binding	180000
$J$15	Octane Rating, Output 2	150000	$J$15>=$L$15	Not Binding	150000
$J$16	Vapour Pressure, Output 1	0	$J$16<=$L$16	Binding	0
$J$17	Vapour Pressure, Output 2	0	$J$17<=$L$17	Binding	0
$J$6	Available, Input 1	25000	$J$6<=$L$6	Binding	0
$J$7	Available, Input 2	23000	$J$7<=$L$7	Not Binding	37000
$J$8	Minimum Prod., Output 1	18000	$J$8>=$L$8	Not Binding	3000
$J$9	Minimum Prod., Output 2	30000	$J$9>=$L$9	Binding	0

In summary the solution is:

$$I_1^* = 25,000$$

$$
\begin{aligned}
I_2^* &= 23,000 \\
X_1^* &= 18,000 \\
X_2^* &= 30,000 \\
U_{1,1}^* &= 15,000 \\
U_{1,2}^* &= 10,000 \\
U_{2,1}^* &= 3,000 \\
U_{2,2}^* &= 20,000
\end{aligned}
$$

The optimal solution uses all the input 1 available, and produces the minimum requirement for output 2. The objective function value is $1,531,000.

The recommendation is to purchase 25,000 cubic metres (or 25,000,000 litres) of input 1, of which 15,000 cubic metres goes into output 1 and 10,000 into output 2, and purchase 23,000 cubic metres of input 2, of which 3,000 goes into output 1, and 20,000 goes into output 2, thereby producing a total of 18,000 cubic metres of output 1 and 30,000 cubic metres of output 2, for a contribution to profit of $1,531,000.

3.2 Scheduling

The example below on scheduling police constables covers this subject, but we also present for optional use a more complex example involving the scheduling of telephone operators.

3.2.1 Scheduling of Police Constables

Description

Members of the Constabulary work twelve hour shifts, beginning at midnight, 3 a.m, 6 a.m, 9 a.m., noon, 3 p.m., 6 p.m., or 9 p.m. On a 24-hour clock basis, we would say that these shifts begin at 00, 03, 06, 09, 12, 15, 18, and 21 hours. These eight shifts are displayed in Figure 3.1.

Since crime and traffic depend on the time of day, so does the minimum number of constables needed:

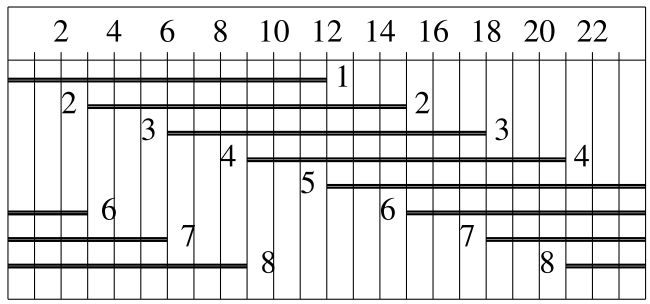


Figure 3.1: Constable Shifts on a 24 Hour Day

Time of Day	Minimum Number of Constables
00 – 03	195
03 – 06	100
06 – 09	160
09 – 12	110
12 – 15	115
15 – 18	135
18 – 21	120
21 – 24	160

Model to Minimize the Number of Constables

To keep this problem simple, let's suppose that all constables are paid the same, and since each works twelve hours, it suffices to minimize the total number of constables over the course of the day, subject to meeting the requirements. Note that the optimal plan may mean that at some times of the day, we might have more constables working than are needed. Let's also assume that each day is the same.

Let $X_i =$ the number of constables who work on shift i $(i = 1, \ldots, 8)$, where shift 1 is the period 00 – 12, shift 2 is 03 – 15, and so on, with shift 8 being 21 – 09. Equivalently, shift i $(i = 1, \ldots, 8)$ is as shown on Figure 3.1.

The objective is to minimize the total number of constables, which is:

$$X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8$$

or

$$\sum_{i=1}^{8} X_i$$

In the first period of the day, from hours 00 to 03, the constables who are working are those who started at midnight (X_1) and also all those who started work on the *previous day* at hour 15 (X_6), hour 18 (X_7), or hour 21 (X_8). Hence the total number of constables working from hours 00 to 03 is $X_1 + X_6 + X_7 + X_8$. This sum must be at least the required number of constables in that time period, which is 195. Hence the constraint for hours 00 to 03 is:

$$X_1 + X_6 + X_7 + X_8 \geq 195$$

.

In the second period of the day, from hours 03 to 06, we gain the workers who have just begun their shift (X_2), but lose those who began at hour 15 the previous day (X_6). Therefore the number of constables on duty from hours 03 to 06 is $X_1 + X_2 + X_7 + X_8$. Since we need a minimum of 100 constables during that time, we require that:

$$X_1 + X_2 + X_7 + X_8 \geq 100$$

There will be eight constraints in total. At the end, we need to note that the solution must be integer. We could write that all variables must be ≥ 0 and integer, or that they must be $\in \{0, 1, 2, \ldots\}$.

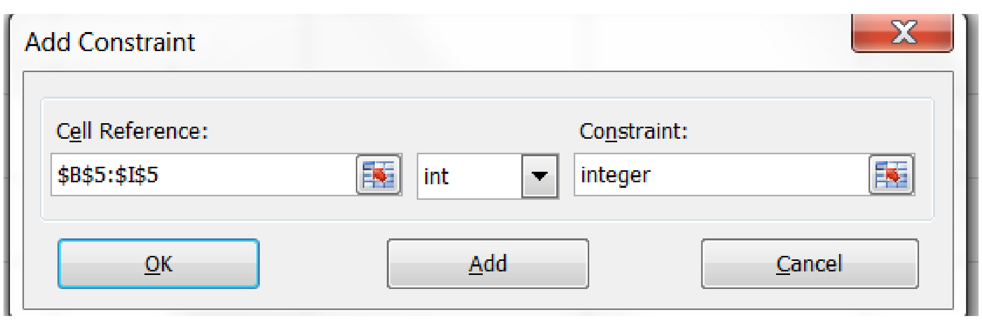
$$\begin{array}{rlcl}
\text{minimize} & \displaystyle\sum_{i=1}^{8} X_i & & \\
\text{subject to} & & & \\
00 - 03 & X_1 + X_6 + X_7 + X_8 & \geq & 195 \\
03 - 06 & X_1 + X_2 + X_7 + X_8 & \geq & 100 \\
06 - 09 & X_1 + X_2 + X_3 + X_8 & \geq & 160 \\
09 - 12 & X_1 + X_2 + X_3 + X_4 & \geq & 110 \\
12 - 15 & X_2 + X_3 + X_4 + X_5 & \geq & 115 \\
15 - 18 & X_3 + X_4 + X_5 + X_6 & \geq & 135 \\
18 - 21 & X_4 + X_5 + X_6 + X_7 & \geq & 120 \\
21 - 24 & X_5 + X_6 + X_7 + X_8 & \geq & 160
\end{array}$$

$$\text{all } X_i \in \{0, 1, 2, \ldots\}$$

Every term in the objective function has a coefficient of 1,[6] hence this model is entered into Excel as:

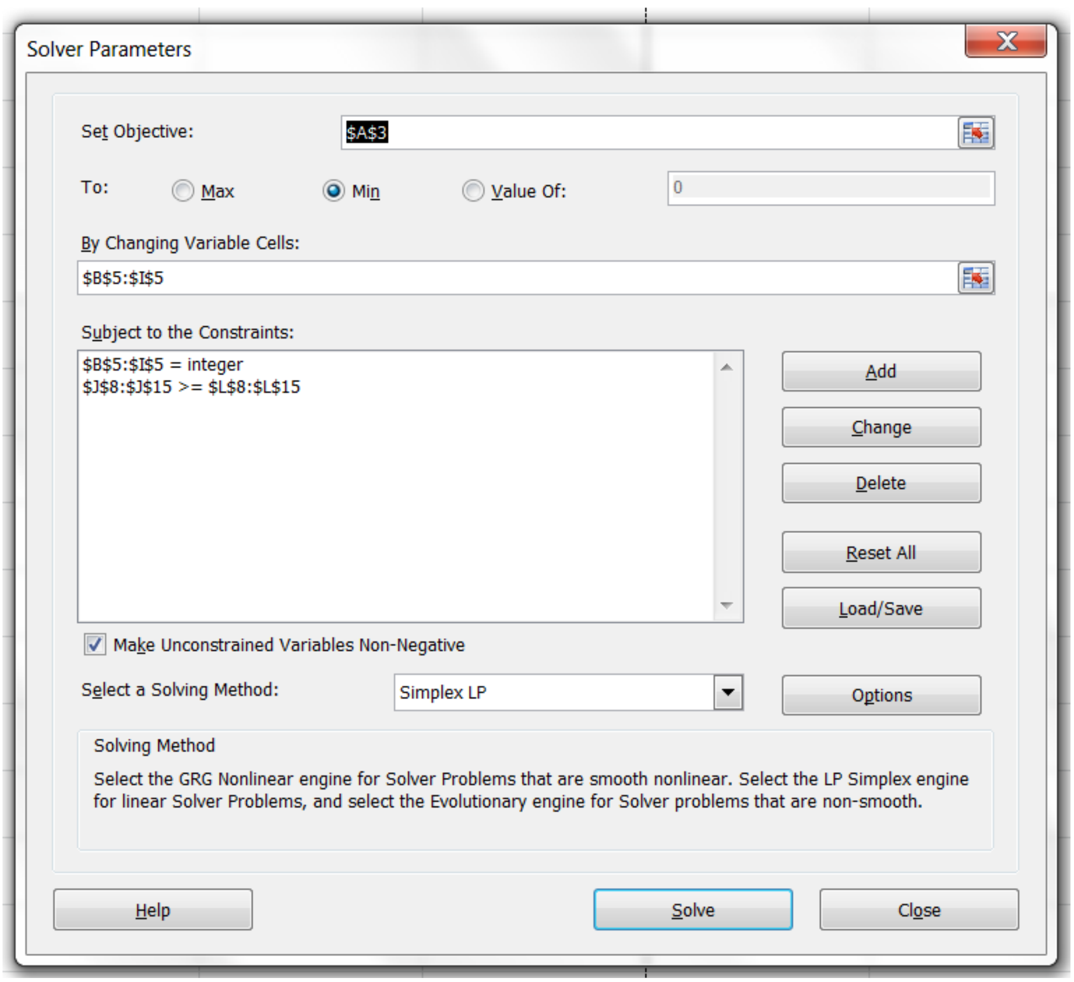
	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	OFV	**Minimal Constable Staffing Model**										
3	0	X1	X2	X3	X4	X5	X6	X7	X8			
4	Minimize	1	1	1	1	1	1	1	1			
5	Constables											
6												
7	Time Periods											RHS
8	00 to 03	1					1	1	1	0	>=	195
9	03 to 06	1	1					1	1	0	>=	100
10	06 to 09	1	1	1					1	0	>=	160
11	09 to 12	1	1	1	1					0	>=	110
12	12 to 15		1	1	1	1				0	>=	115
13	15 to 18			1	1	1	1			0	>=	135
14	18 to 21				1	1	1	1		0	>=	120
15	21 to 24					1	1	1	1	0	>=	160

In cell J8, we enter `=SUMPRODUCT(B$5:I$5,B8:J8)`, and then copy this into the range `J8:J15`. The only thing that is new from what we saw in the previous chapter is that we must tell the Solver that the variables must be integer. To do this we use the *Add Constraint* dialog box, declaring the range `B5:I5` to be “int” (middle box), which causes the word “integer” to be automatically entered into the right box.



[6]In this example, we could have omitted the 1’s in row 4, and simply have entered =SUM(B3:I3) into cell A3, but for consistency with the other Excel models we kept the 1’s and used the **SUMPRODUCT** function.

Actually, this particular example is naturally integer, meaning that we would have obtained an integer solution even without this declaration, but there‘s no way to know this in advance. The entire *Solver Parameters* dialog box is:



Solving we obtain:

	A	B	C	D	E	F	G	H	I	J	K	L
1												
2	OFV	**Minimal Constable Staffing Model**										
3	310	X1	X2	X3	X4	X5	X6	X7	X8			
4	Minimize	1	1	1	1	1	1	1	1			
5	Constables	150	0	0	0	115	35	0	10			
6												
7	Time Periods											RHS
8	00 to 03	1					1	1	1	195	>=	195
9	03 to 06	1	1					1	1	160	>=	100
10	06 to 09	1	1	1					1	160	>=	160
11	09 to 12	1	1	1	1					150	>=	110
12	12 to 15		1	1	1	1				115	>=	115
13	15 to 18			1	1	1	1			150	>=	135
14	18 to 21				1	1	1	1		150	>=	120
15	21 to 24					1	1	1	1	160	>=	160

Hence, 150 constables begin their shift at midnight, followed by 115 at noon, another 35 at 3 p.m,, and finally another 10 at 9 p.m. Though this is the optimal solution, at some times in the day there are many more constables on duty than are required, especially from 9 a.m. to noon. This would be a good time to let the constables attend to other things, such as medical appointments.

Scheduling Constables with Shift Premiums

In the previous section, since all shifts had equal pay, we simply minimized the total number of constables. Now suppose that all constables earn a base rate of \$45 per hour, but are paid a bonus of \$9 per hour when working from midnight to 6 a.m. Now the objective function will be in dollars, instead of the number of constables.

One way to approach this would be to calculate what each constable is paid on each of the eight shifts. Everyone is paid a base rate of \$45(12) = \$540. Since shifts 3, 4, and 5 all work outside of the bonus period, their coefficients in the objective function will all be 540. Those who work on shifts 2 and 6 work three hours in the bonus period, and so are paid an extra \$9(3) = \$27 for a total of \$567.

Finally, those who work on shifts 1, 7, and 8 work six hours in the bonus period, and so they make an extra \$9(6) = \$54 for a total of \$594. Using this approach, the objective function would be:

$$\text{minimize} \quad 594X_1 + 567X_2 + 540X_3 + 540X_4 + 540X_5 + 567X_6 + 594X_7 + 594X_8$$

While this is correct, there are three disadvantages to doing things this way. First, there is a loss of transparency, because the \$45 and the \$9 are not visible to someone looking at either the algebraic model or the spreadsheet model. Secondly, if the \$45 or the \$9 were to change, it would involve the recalculation of all the objective function coefficients. Thirdly, the calculation of these coefficients is a potential source of error.

The other approach is to define two new variables. Since the \$45 per hour is always paid, it makes sense to define a variable for the total number of hours worked; we let this variable be H_1. We define H_2 to be the number of hours worked for which the bonus is paid. The objective function is simply

$$\text{minimize} \qquad 45H_1 + 9H_2$$

We need to add two constraints to the model. The first defines the total number of hours:

$$H_1 = 12X_1 + 12X_2 + 12X_3 + 12X_4 + 12X_5 + 12X_6 + 12X_7 + 12X_8$$

Equivalently, we could write:

$$H_1 = 12\left(X_1 + X_2 + X_3 + X_4 + X_5 + X_6 + X_7 + X_8\right)$$

or even

$$H_1 = 12\left(\sum_{i=1}^{8} X_i\right)$$

We put the variables on the left. We can either do this with one positive coefficient and eight negative ones, or one negative coefficient and eight positive ones. Doing the latter we obtain:

$$-H_1 + 12\left(\sum_{i=1}^{8} X_i\right) = 0$$

As we have said, those who work on shifts 1, 7, and 8 earn the bonus for 6 hours (midnight to 6 a.m.), and those who work on shift 2 and 6 earn it for 3 hours. Hence we have

$$H_2 \quad = \quad 6X_1 + 3X_2 + 3X_6 + 6X_7 + 6X_8$$

Putting all the variables in standard form we obtain:

$$-H_2 + 6X_1 + 3X_2 + 3X_6 + 6X_7 + 6X_8 = 0$$

Variables H_1 and H_2 will turn out to be integer, because they are obtained by multiplying integer variables by integers. However, there is no need to declare them as such.[7] Hence the new algebraic model is:

$$\begin{array}{rrcl}
\text{minimize} & 45H_1 + 9H_2 & & \\
\text{subject to} & & & \\
\text{Balance on Hours} & & & \\
\text{Total} & -H_1 + 12\left(\sum_{i=1}^{8} X_i\right) & = & 0 \\
\text{Bonus Pay} & -H_2 + 6X_1 + 3X_2 + 3X_6 + 6X_7 + 6X_8 & = & 0 \\
\text{Staffing by Time of Day} & & & \\
00-03 & X_1 + X_6 + X_7 + X_8 & \geq & 195 \\
03-06 & X_1 + X_2 + X_7 + X_8 & \geq & 100 \\
06-09 & X_1 + X_2 + X_3 + X_8 & \geq & 160 \\
09-12 & X_1 + X_2 + X_3 + X_4 & \geq & 110 \\
12-15 & X_2 + X_3 + X_4 + X_5 & \geq & 115 \\
15-18 & X_3 + X_4 + X_5 + X_6 & \geq & 135 \\
18-21 & X_4 + X_5 + X_6 + X_7 & \geq & 120 \\
21-24 & X_5 + X_6 + X_7 + X_8 & \geq & 160 \\
 & & & \\
 & H_1, H_2 \geq 0 & & \\
 & \text{all } X_i \in \{0, 1, 2, \ldots\} & &
\end{array}$$

In the Excel spreadsheet which follows the formula in cell B3 is `=SUMPRODUCT(C4:L4,C5:L5)`, and the formula in cell M7 which is copied into the range M7:M16 is `=SUMPRODUCT($C$5:$L$5,C7:L7)`.

[7] Indeed if constables could work say 4.5 hours at the bonus rate, then variables H_1 and H_2 would have to be continuous rather than integer.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1															
2		OFV	**Minimal Cost Constable Staffing Model**												
3		$0.00	H1	H2	X1	X2	X3	X4	X5	X6	X7	X8			
4		Minimize	45	9	0	0	0	0	0	0	0	0			
5															
6		Constraints													
7	Balance	Total	-1	0	12	12	12	12	12	12	12	12	0	=	0
8	on Hours	Bonus Pay	0	-1	6	3				3	6	6	0	=	0
9		00 to 03			1					1	1	1	0	>=	195
10		03 to 06			1	1					1	1	0	>=	100
11	Time	06 to 09			1	1	1					1	0	>=	160
12	Periods	09 to 12			1	1	1	1					0	>=	110
13		12 to 15				1	1	1	1				0	>=	115
14		15 to 18					1	1	1	1			0	>=	135
15		18 to 21						1	1	1	1		0	>=	120
16		21 to 24							1	1	1	1	0	>=	160

We minimize cell B3, with M7:M8 = O7:O8, M9:M16 $\geq$ O9:O16, and E5:L5 declared to be integer. The solution is:

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O
1															
2		OFV	**Minimal Cost Constable Staffing Model**												
3		$175,365.00	H1	H2	X1	X2	X3	X4	X5	X6	X7	X8			
4		Minimize	45	9	0	0	0	0	0	0	0	0			
5			3720	885	75	0	115	0	0	95	25	0			
6		Constraints													
7	Balance	Total	-1	0	12	12	12	12	12	12	12	12	0	=	0
8	on Hours	Bonus Pay	0	-1	6	3				3	6	6	0	=	0
9		00 to 03			1					1	1	1	195	>=	195
10		03 to 06			1	1					1	1	100	>=	100
11	Time	06 to 09			1	1	1					1	190	>=	160
12	Periods	09 to 12			1	1	1	1					190	>=	110
13		12 to 15				1	1	1	1				115	>=	115
14		15 to 18					1	1	1	1			210	>=	135
15		18 to 21						1	1	1	1		120	>=	120
16		21 to 24							1	1	1	1	120	>=	160

The optimal solution is to have 75 constables beginning to work at midnight, 115 beginning at 6 a.m., 95 beginning at 3 p.m., and 25 beginning at 6 p.m. The total of 310 constables work 3,720 hours, of which 885 hours are paid the night-time bonus, with a total daily cost of $175,365.

3.2.2 Telephone Operator Problem (Optional)

Real-life employee scheduling problems are much more complex than the preceding example. In the following example, we make the need for employees based on each hour of a 24-hour day. Also, we add one new factor, that of accounting for lunch breaks. These two changes lead to a much larger model.

Description

The collective bargaining agreement between a telephone company and the union which represents its employees states that each operator works an eight hour shift, with a one-hour break during either the fourth or the fifth hour of the shift. An employee's shift can begin at midnight, 2 a.m., 4 a.m., 6 a.m., 8 a.m., 10 a.m., noon, 2 p.m., 4 p.m., 6 p.m., 8 p.m., or 10 p.m. The company has the right to decide how many persons will begin their shifts at these specified times, and how many within each shift take an early or late break. The employees can bid on these shifts according to seniority.

The telephone company has set a standard for operator response time. This standard, combined with the anticipated customer demand which varies according to the time of day, gives rise to a minimum number of operators needed each hour. On a 24 hour clock basis the requirements are:

Hour of the Day	1	2	3	4	5	6	7	8
Minimum Number of Operators	5	3	2	2	2	3	4	4
Hour of the Day	9	10	11	12	13	14	15	16
Minimum Number of Operators	7	12	15	20	25	24	18	16
Hour of the Day	17	18	19	20	21	22	23	24
Minimum Number of Operators	20	10	8	6	6	5	5	5

To keep the context simple, we will assume that each day's requirements are the same. Also, we will ignore the fact that to more accurately reflect a real-world problem, an operator would have to be scheduled for a week with two days off; we will simply treat this as a one-day problem.

Given management's obligations and flexibility as allowed by the collective bargaining agreement, and given the market driven demand for telephone operators, what is the minimum number of operators needed each day? If we can answer this question, we will also know how to minimize the wastage of employees resulting from more employees being at work than are required. Neither of these two related issues requires a knowledge of what the hourly rate of pay is. Of course, minimizing cost would be the objective if shift premiums (e.g. for night shifts) were to be paid. An example of such a situation appears at the end of this section.

Formulation

In problems such as this it is useful to index the shifts. One way is to ignore the lunch breaks and just consider the hours during which an employee is on the telephone company's premises. Another approach would consider two employees who commence work at the same time but who take different lunch breaks to be working distinct shifts. The first approach yields twelve shifts, the second yields twenty-four. Adopting the first approach the shifts are:

Shift #	Hours
1	midnight - 8 a.m.
2	2 a.m. - 10 a.m.
3	4 a.m. - noon
4	6 a.m. - 2 p.m.
5	8 a.m. - 4 p.m.
6	10 a.m. - 6 p.m.
7	noon - 8 p.m.
8	2 p.m. - 10 p.m.
9	4 p.m. - midnight
10	6 p.m. - 2 a.m.
11	8 p.m. - 4 a.m.
12	10 p.m. - 6 a.m.

Let $X_i =$ the number of operators who work on shift i, and who take an early break ($i = 1, 2, \ldots, 12$).

Let $Y_i =$ the number of operators who work on shift i, and who take a late break ($i = 1, 2, \ldots, 12$).

In order to answer the question of determining the minimum number of operators required, the appropriate objective function is:

$$\text{minimize} \sum_{i=1}^{12} (X_i + Y_i).$$

There is a constraint for each hour of the day. Each constraint will ensure that the actual number of operators working during a particular hour will meet or exceed the minimal staffing requirement during that hour. For example, in the first hour of the day (midnight to 1 a.m.), the employees working are those who began at midnight (of whom there are $X_1 + Y_1$), plus those who began to work at any time on or after 6 p.m. of the previous day, except for those who began at 8 p.m. and who are taking a late lunch break. Hence the total number of employees working from midnight to 1 a.m. is $X_1 + Y_1 + X_{10} + Y_{10} + X_{11} + X_{12} + Y_{12}$. (Note the exclusion of Y_{11} from this list, since these employees are on their break.) The minimal staffing requirement for the first hour is five operators, hence we require that:

$$X_1 + Y_1 + X_{10} + Y_{10} + X_{11} + X_{12} + Y_{12} \geq 5$$

In the second hour, the employees who were on their break have returned to work (Y_{11}), and the employees who began at 10 p.m. who take an early break (there are

Legend: Thick lines – Working; ⊗ – Break

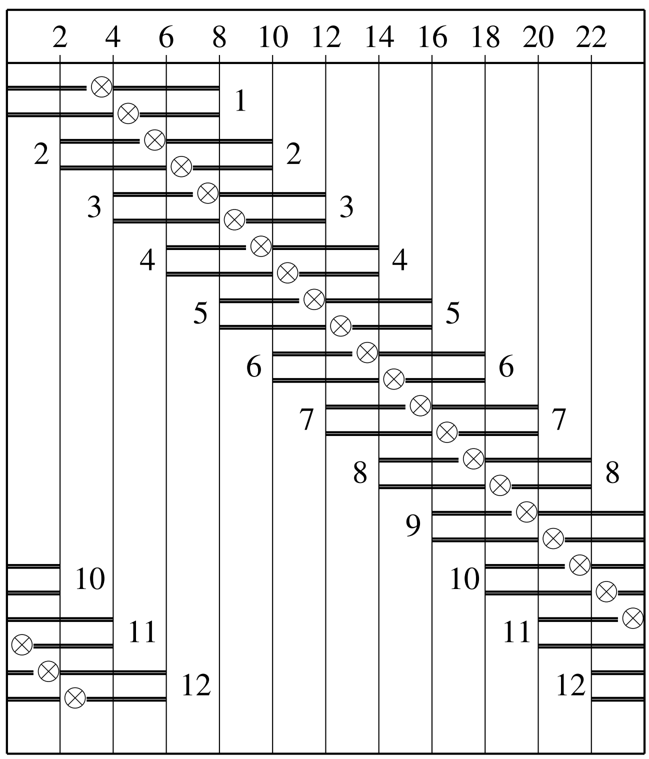


Figure 3.2: Operator Shifts on a 24 Hour Day

X_{12} of them) are now on their break. Since at least three operators are required in the second hour, the second constraint is:

$$X_1 + Y_1 + X_{10} + Y_{10} + X_{11} + Y_{11} + Y_{12} \geq 3$$

In the third hour, the employees who began at 6 p.m. ($X_{10} + Y_{10}$) have finished work, and $X_2 + Y_2$ have just begun. Making an adjustment for those who end or begin a break at 3 a.m., the third hour constraint is:

$$X_1 + Y_1 + X_2 + Y_2 + X_{11} + Y_{11} + X_{12} \geq 2$$

We could continue to determine the other twenty-one constraints in this manner, but it is helpful to draw a diagram to help understand how the shifts look. This diagram is shown in Figure 3.2.

Each column of this figure gives the overlap of the workers for a particular hour. We see that for each hour there are seven sets of operators. From this figure we obtain the other twenty-one constraints.

As with all the models that we have seen so far, there will be non-negativity restrictions on the variables. In addition, because these variables represent numbers of people, each of them must be integer. We therefore say that each variable must be contained in the set of positive integers, either by writing $\in \{0, 1, 2, 3, \ldots\}$, or by writing ≥ 0 and $\in I$.

Combining the objective function, the twenty-four constraints, and the non-negativity and integer restrictions yields:

$$
\begin{array}{rrcr}
\text{minimize} & \displaystyle\sum_{i=1}^{12}(X_i+Y_i) & & \\
\text{subject to} & & & \\
\text{Staffing in} & & & \\
\text{Hour 1} & X_1+Y_1+X_{10}+Y_{10}+X_{11}+X_{12}+Y_{12} & \geq & 5 \\
\text{Hour 2} & X_1+Y_1+X_{10}+Y_{10}+X_{11}+Y_{11}+Y_{12} & \geq & 3 \\
\text{Hour 3} & X_1+Y_1+X_2+Y_2+X_{11}+Y_{11}+X_{12} & \geq & 2 \\
\text{Hour 4} & Y_1+X_2+Y_2+X_{11}+Y_{11}+X_{12}+Y_{12} & \geq & 2 \\
\text{Hour 5} & X_1+X_2+Y_2+X_3+Y_3+X_{12}+Y_{12} & \geq & 2 \\
\text{Hour 6} & X_1+Y_1+Y_2+X_3+Y_3+X_{12}+Y_{12} & \geq & 3 \\
\text{Hour 7} & X_1+Y_1+X_2+X_3+Y_3+X_4+Y_4 & \geq & 4 \\
\text{Hour 8} & X_1+Y_1+X_2+Y_2+Y_3+X_4+Y_4 & \geq & 4 \\
\text{Hour 9} & X_2+Y_2+X_3+X_4+Y_4+X_5+Y_5 & \geq & 7 \\
\text{Hour 10} & X_2+Y_2+X_3+Y_3+Y_4+X_5+Y_5 & \geq & 12 \\
\text{Hour 11} & X_3+Y_3+X_4+X_5+Y_5+X_6+Y_6 & \geq & 15 \\
\text{Hour 12} & X_3+Y_3+X_4+Y_4+Y_5+X_6+Y_6 & \geq & 20 \\
\text{Hour 13} & X_4+Y_4+X_5+X_6+Y_6+X_7+Y_7 & \geq & 25 \\
\text{Hour 14} & X_4+Y_4+X_5+Y_5+Y_6+X_7+Y_7 & \geq & 24 \\
\text{Hour 15} & X_5+Y_5+X_6+X_7+Y_7+X_8+Y_8 & \geq & 18 \\
\text{Hour 16} & X_5+Y_5+X_6+Y_6+Y_7+X_8+Y_8 & \geq & 16 \\
\text{Hour 17} & X_6+Y_6+X_7+X_8+Y_8+X_9+Y_9 & \geq & 20 \\
\text{Hour 18} & X_6+Y_6+X_7+Y_7+Y_8+X_9+Y_9 & \geq & 10 \\
\text{Hour 19} & X_7+Y_7+X_8+X_9+Y_9+X_{10}+Y_{10} & \geq & 8 \\
\text{Hour 20} & X_7+Y_7+X_8+Y_8+Y_9+X_{10}+Y_{10} & \geq & 6 \\
\text{Hour 21} & X_8+Y_8+X_9+X_{10}+Y_{10}+X_{11}+Y_{11} & \geq & 6 \\
\text{Hour 22} & X_8+Y_8+X_9+Y_9+Y_{10}+X_{11}+Y_{11} & \geq & 5 \\
\text{Hour 23} & X_9+Y_9+X_{10}+X_{11}+Y_{11}+X_{12}+Y_{12} & \geq & 5 \\
\text{Hour 24} & X_9+Y_9+X_{10}+Y_{10}+Y_{11}+X_{12}+Y_{12} & \geq & 5
\end{array}
$$

all variables must be ≥ 0 and $\in I$

Solution of the Model

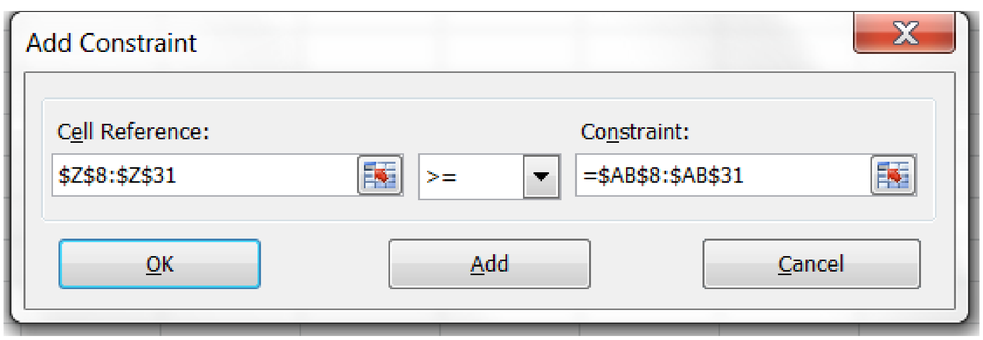
We put this formulation onto a spreadsheet. Here are a few points relevant to this example:

1. We can write the formula in cell A3 as `=SUMPRODUCT(B4:Y4,B5:Y5)`

However, since every number in the range B4:Y4 is a 1, the formula `=SUM(B5:Y5)` would work as well.

2. We enter `=SUMPRODUCT($B$5:$Y$5,B8:Y8)` into cell Z8, and copy this into the range Z8:Z31.

3. In the Solver, since all constraints are of the $\geq$ form, we add them all at once:



Transforming the algebraic model to a spreadsheet model, and then optimizing using the Solver we obtain:

	A	B	C	D	E	F	G	H	I	J	K	L	M
2	OFV			**Minimal Telephone Operator Staffing Model**									
3	37	X1	Y1	X2	Y2	X3	Y3	X4	Y4	X5	Y5	X6	Y6
4	Minimize	1	1	1	1	1	1	1	1	1	1	1	1
5	Operators	0	0	0	0	0	0	0	7	2	3	4	6
6													
7	Constraints												
8	Hour 1	1	1										
9	Hour 2	1	1										
10	Hour 3	1	1	1	1								
11	Hour 4		1	1	1								
12	Hour 5	1		1	1	1	1						
13	Hour 6	1	1		1	1	1						
14	Hour 7	1	1	1		1	1	1	1				
15	Hour 8	1	1	1	1		1	1	1				
16	Hour 9			1	1	1		1	1	1	1		
17	Hour 10			1	1	1	1		1	1	1		
18	Hour 11					1	1	1		1	1	1	1
19	Hour 12					1	1	1	1		1	1	1
20	Hour 13							1	1	1		1	1
21	Hour 14							1	1	1	1		1
22	Hour 15									1	1	1	
23	Hour 16									1	1	1	1
24	Hour 17											1	1
25	Hour 18											1	1
26	Hour 19												
27	Hour 20												
28	Hour 21												
29	Hour 22												
30	Hour 23												
31	Hour 24												

	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB
2															
3	X7	Y7	X8	Y8	X9	Y9	X10	Y10	X11	Y11	X12	Y12			
4	1	1	1	1	1	1	1	1	1	1	1	1			
5	6	0	0	3	1	0	1	0	1	0	2	1			
6															
7															RHS
8							1	1	1		1	1	5	>=	5
9							1	1	1	1		1	3	>=	3
10									1	1	1		3	>=	2
11									1	1	1	1	4	>=	2
12											1	1	3	>=	2
13											1	1	3	>=	3
14													7	>=	4
15													7	>=	4
16													12	>=	7
17													12	>=	12
18													15	>=	15
19													20	>=	20
20	1	1											25	>=	25
21	1	1											24	>=	24
22	1	1	1	1									18	>=	18
23		1	1	1									18	>=	16
24	1		1	1	1	1							20	>=	20
25	1	1		1	1	1							20	>=	10
26	1	1	1		1	1	1	1					8	>=	8
27	1	1	1	1		1	1	1					10	>=	6
28			1	1	1		1	1	1	1			6	>=	6
29			1	1	1	1		1	1	1			5	>=	5
30					1	1	1		1	1	1	1	6	>=	5
31					1	1	1	1		1	1	1	5	>=	5

This example is naturally integer, so whether or not we invoked the ***int*** command in the Solver we obtain an integer solution. The algebraic solution is: OFV = 37, $Y_4 = 7$, $X_5 = 2$, $Y_5 = 3$, $X_6 = 4$, $Y_6 = 6$, $X_7 = 6$, $Y_8 = 3$, $X_9 = 1$, $X_{10} = 1$, $X_{11} = 1$, $X_{12} = 1$, and $Y_{12} = 1$. Note that there may be multiple optima, meaning that there's another solution with different values for the variables, but with the same objective function value of 37.

The managerial solution is: We require 37 operators in total. At 6 a.m., seven persons begin their shift, and all of these take a late lunch. Five people begin at 8 a.m., two with an early lunch and three with a late lunch. Ten people start at 10 a.m., four with an early lunch and six with a late lunch. Six people begin at 12 noon, and all of these take an early lunch. Three people begin at 2 p.m., and all of these take a late lunch. One person begins at each of 4 p.m., 6 p.m., and 8 a.m. and all these persons take an early lunch. Finally three people begin at 10 p.m., with two on an early lunch and one on a late lunch.

It is interesting to examine the surplus ("Slack" on the Excel Solver) on each constraint. From the Answer Report we see that ten of the twenty-four are strictly positive, with the largest being 10 in the 18th hour (5 to 6 p.m.). If occasionally some of the employees need an hour off for medical exams, safety talks and so on, this would be a good time for it. The sum of the surpluses, which is 32, means that there is a total of 32 person-hours paid for but not required each day. This is out of a total of $37 \times 8 = 296$ person-hours each day, hence the "waste" is about 10.8%. Knowing this figure gives the union and management information about the benefits of more flexible work-rules.

An Alternate Objective

Suppose that the constraints are the same as before, but now the objective is to minimize the cost of labour, where each operator is paid a base rate of \$30 per hour in wages and benefits, and a night shift premium of \$6 per hour in wages and benefits between 10 p.m. and 6 a.m. (We assume that the break hour is also paid.)

Since the \$30 per hour is always paid, it makes sense to define H_1 as the total number of hours worked. We define H_2 to be the number of hours worked for which the bonus is paid. The objective function is simply

$$\text{minimize} \qquad 30H_1 + 6H_2$$

We need to add two constraints to the model. The first defines the total number of hours:

$$H_1 = 8\left(\sum_{i=1}^{12}(X_i + Y_i)\right)$$

We put the variables on the left. We can either do this with one positive coefficient and twenty-four negative ones, or one negative coefficient and twenty-four positive ones. Doing the latter we obtain:

$$-H_1 + 8\left(\sum_{i=1}^{12}(X_i + Y_i)\right) = 0$$

The expression for H_2 requires a bit more thought. Those who work on shift 1 earn the bonus for 6 hours (midnight to 6 a.m.); those who work on shift 2 earn it for 4 hours, and so on. Hence we have

$$\begin{aligned} H_2 &= 6X_1 + 6Y_1 + 4X_2 + 4Y_2 + 2X_3 + 2Y_3 \\ &\quad +2X_9 + 2Y_9 + 4X_{10} + 4Y_{10} + 6X_{11} + 6Y_{11} + 8X_{12} + 8Y_{12} \end{aligned}$$

Putting all the variables in standard form we obtain:

$$\begin{aligned} -H_2 + 6X_1 + 6Y_1 + 4X_2 + 4Y_2 + 2X_3 + 2Y_3 & \\ +2X_9 + 2Y_9 + 4X_{10} + 4Y_{10} + 6X_{11} + 6Y_{11} + 8X_{12} + 8Y_{12} &= 0 \end{aligned}$$

Variables H_1 and H_2 need not be integer, though as before all the X and Y variables must be integer. Hence the new algebraic model is:

$$
\begin{array}{rrcl}
\text{minimize} & 30H_1 + 6H_2 & & \\
\text{subject to} & & & \\
\text{Balance on } H_1 & -H_1 + 8\left(\sum_{i=1}^{12}(X_i + Y_i)\right) & = & 0 \\
\text{Balance on } H_2 & -H_2 + 6X_1 + 6Y_1 + 4X_2 + 4Y_2 + 2X_3 & & \\
 & +2Y_3 + 2X_9 + 2Y_9 + 4X_{10} + 4Y_{10} & & \\
 & +6X_{11} + 6Y_{11} + 8X_{12} + 8Y_{12} & = & 0 \\
\text{Staffing in} & & & \\
\text{Hour 1} & X_1 + Y_1 + X_{10} + Y_{10} + X_{11} + X_{12} + Y_{12} & \geq & 5 \\
\text{Hour 2} & X_1 + Y_1 + X_{10} + Y_{10} + X_{11} + Y_{11} + Y_{12} & \geq & 3 \\
\text{Hour 3} & X_1 + Y_1 + X_2 + Y_2 + X_{11} + Y_{11} + X_{12} & \geq & 2 \\
\text{Hour 4} & Y_1 + X_2 + Y_2 + X_{11} + Y_{11} + X_{12} + Y_{12} & \geq & 2 \\
\text{Hour 5} & X_1 + X_2 + Y_2 + X_3 + Y_3 + X_{12} + Y_{12} & \geq & 2 \\
\text{Hour 6} & X_1 + Y_1 + Y_2 + X_3 + Y_3 + X_{12} + Y_{12} & \geq & 3 \\
\text{Hour 7} & X_1 + Y_1 + X_2 + X_3 + Y_3 + X_4 + Y_4 & \geq & 4 \\
\text{Hour 8} & X_1 + Y_1 + X_2 + Y_2 + Y_3 + X_4 + Y_4 & \geq & 4 \\
\text{Hour 9} & X_2 + Y_2 + X_3 + X_4 + Y_4 + X_5 + Y_5 & \geq & 7 \\
\text{Hour 10} & X_2 + Y_2 + X_3 + Y_3 + Y_4 + X_5 + Y_5 & \geq & 12 \\
\text{Hour 11} & X_3 + Y_3 + X_4 + X_5 + Y_5 + X_6 + Y_6 & \geq & 15 \\
\text{Hour 12} & X_3 + Y_3 + X_4 + Y_4 + Y_5 + X_6 + Y_6 & \geq & 20 \\
\text{Hour 13} & X_4 + Y_4 + X_5 + X_6 + Y_6 + X_7 + Y_7 & \geq & 25 \\
\text{Hour 14} & X_4 + Y_4 + X_5 + Y_5 + Y_6 + X_7 + Y_7 & \geq & 24 \\
\text{Hour 15} & X_5 + Y_5 + X_6 + X_7 + Y_7 + X_8 + Y_8 & \geq & 18 \\
\text{Hour 16} & X_5 + Y_5 + X_6 + Y_6 + Y_7 + X_8 + Y_8 & \geq & 16 \\
\text{Hour 17} & X_6 + Y_6 + X_7 + X_8 + Y_8 + X_9 + Y_9 & \geq & 20 \\
\text{Hour 18} & X_6 + Y_6 + X_7 + Y_7 + Y_8 + X_9 + Y_9 & \geq & 10 \\
\text{Hour 19} & X_7 + Y_7 + X_8 + X_9 + Y_9 + X_{10} + Y_{10} & \geq & 8 \\
\text{Hour 20} & X_7 + Y_7 + X_8 + Y_8 + Y_9 + X_{10} + Y_{10} & \geq & 6 \\
\text{Hour 21} & X_8 + Y_8 + X_9 + X_{10} + Y_{10} + X_{11} + Y_{11} & \geq & 6 \\
\text{Hour 22} & X_8 + Y_8 + X_9 + Y_9 + Y_{10} + X_{11} + Y_{11} & \geq & 5 \\
\text{Hour 23} & X_9 + Y_9 + X_{10} + X_{11} + Y_{11} + X_{12} + Y_{12} & \geq & 5 \\
\text{Hour 24} & X_9 + Y_9 + X_{10} + Y_{10} + Y_{11} + X_{12} + Y_{12} & \geq & 5 \\
 & & & \\
 & H_1, H_2 \geq 0 & & \\
 & \text{all } X_i \text{ and } Y_i \in \{0, 1, 2, 3, \ldots\} & &
\end{array}
$$

This model is put into spreadsheet form and is optimized using the Solver. We know that we must declare the X_i and Y_i variables to be ***int***, but first we find out what happens if do not do this.

	A	B	C	D	E	F	G	H	I	J	K	L	M	N
1														
2	OFV	**Cost Minimization Telephone Operator Staffing Model**												
3	$9,080.00	H1	H2	X1	Y1	X2	Y2	X3	Y3	X4	Y4	X5	Y5	X6
4	Minimize	30	6	0	0	0	0	0	0	0	0	0	0	0
5	Operators	296	33.3	0	0	0	0	0	0	0	4	5.33	2.67	3.67
6														
7	Constraints													
8	Balance H1	-1		8	8	8	8	8	8	8	8	8	8	8
9	Balance H2		-1	6	6	4	4	2	2					
10	Hour 1			1	1									
11	Hour 2			1	1									
12	Hour 3			1	1	1	1							
13	Hour 4				1	1	1							
14	Hour 5			1		1	1	1	1					
15	Hour 6			1	1		1	1	1					
16	Hour 7			1	1	1		1	1	1	1			
17	Hour 8			1	1	1	1		1	1	1			
18	Hour 9					1	1	1		1	1	1	1	
19	Hour 10					1	1	1	1		1	1	1	
20	Hour 11							1	1	1		1	1	1
21	Hour 12							1	1	1	1		1	1
22	Hour 13									1	1	1		1
23	Hour 14									1	1	1	1	
24	Hour 15											1	1	1
25	Hour 16											1	1	1
26	Hour 17													1
27	Hour 18													1
28	Hour 19													
29	Hour 20													
30	Hour 21													
31	Hour 22													
32	Hour 23													
33	Hour 24													

	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA	AB	AC	AD
1																
2																
3	Y6	X7	Y7	X8	Y8	X9	Y9	X10	Y10	X11	Y11	X12	Y12			
4	0	0	0	0	0	0	0	0	0	0	0	0	0			
5	9.67	2.33	0	3.67	0.33	0	0.33	1.33	0.33	0.33	0	1.67	1.33			
6																
7																RHS
8	8	8	8	8	8	8	8	8	8	8	8	8	8	0	=	0
9						2	2	4	4	6	6	8	8	-0	=	0
10								1	1	1		1	1	5	>=	5
11								1	1	1	1		1	3.3	>=	3
12										1	1	1		2	>=	2
13										1	1	1	1	3.3	>=	2
14												1	1	3	>=	2
15												1	1	3	>=	3
16														4	>=	4
17														4	>=	4
18														12	>=	7
19														12	>=	12
20	1													21	>=	15
21	1													20	>=	20
22	1	1	1											25	>=	25
23	1	1	1											24	>=	24
24		1	1	1	1									18	>=	18
25	1		1	1	1									25	>=	16
26	1	1		1	1	1	1							20	>=	20
27	1	1	1		1	1	1							16	>=	10
28		1	1	1		1	1	1	1					8	>=	8
29		1	1	1	1		1	1	1					8.3	>=	6
30				1	1	1		1	1	1	1			6	>=	6
31				1	1	1	1		1	1	1			5	>=	5
32						1	1	1		1	1	1	1	5	>=	5
33						1	1	1	1		1	1	1	5	>=	5

We this that this solution contains fractional values for some of the X_i and Y_i variables, which is not what we want. From the Excel Solver we obtain OFV = \$9080.00, $H_1 = 296$, $H_2 = 33\frac{1}{3}$, $X_5 = 5\frac{1}{3}$, $Y_5 = 2\frac{2}{3}$, $X_6 = 3\frac{2}{3}$, $Y_6 = 9\frac{2}{3}$, $X_7 = 2\frac{1}{3}$,

$X_8 = 3\frac{2}{3}$, $Y_8 = \frac{1}{3}$ $Y_9 = \frac{1}{3}$, $X_{10} = 1\frac{1}{3}$, $Y_{10} = \frac{1}{3}$, $X_{11} = \frac{1}{3}$, $X_{12} = 1\frac{2}{3}$, $Y_{12} = 1\frac{1}{3}$, with all other variables being 0.

We use the Solver in Excel to force the variables to have integer values as follows:.

1. Open the Solver, and click on “Add”.

2. The “Add Constraint” dialog box appears, with a blinker in the space below “Cell Reference:”.

3. Use the mouse to highlight the variable cells (in this example, this is `D5:AA5`. The expression `$D$5:$AA$5` will appear in the space.

4. In the middle where the “<=” appears, click on the down arrow to the right, and then click on “int”.

5. The “<=” will be replaced by “int”, and “integer” will appear in the space to the right.

6. Click on “OK”.

7. In the “Solver Parameters” dialog box, `$D$5:$AA$5 = integer` will appear in the “Subject to the Constraints” section.

8. Click on the Solve button.

Doing the above the integer solution is found to be:

	A	B	C	D	E	F	G	H	I	J	K	L	M
3	$9,084.00	H1	H2	X1	Y1	X2	Y2	X3	Y3	X4	Y4	X5	Y5
4	Minimize	30	6	0	0	0	0	0	0	0	0	0	0
5	Operators	296	34	0	0	0	0	1	0	0	5	3	3

	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	AA
3	X6	Y6	X7	Y7	X8	Y8	X9	Y9	X10	Y10	X11	Y11	X12	Y12
4	0	0	0	0	0	0	0	0	0	0	0	0	0	0
5	4	7	6	0	0	2	1	0	1	1	1	0	1	1

The integer declaration causes the OFV to increase from \$9,080 to \$9,084. The optimal values for the decision variables are now $H_1 = 296$, $H_2 = 34$, $X_3 = 1$, $Y_4 = 5$, $X_5 = 3$, $Y_5 = 3$, $X_6 = 4$, $Y_6 = 7$, $X_7 = 6$, $Y_8 = 2$, $X_9 = 1$, $X_{10} = 1$, $Y_{10} = 1$, $X_{11} = 1$, $X_{12} = 1$, $Y_{12} = 1$, all other variables being 0.

The managerial recommendation is: One person begins at 4 a.m. and takes an early lunch break; five persons begin at 6 a.m. and take a late lunch break; six people begin at 8 a.m. three take an early lunch and three take a late lunch; eleven people begin at 10 a.m., four take an early lunch and seven take a late lunch; six people begin at 12 noon, all of whom take an early lunch; two people begin at 2 p.m., both of whom take a late lunch; one person begins at 4 p.m. and takes an early lunch; two persons begin at 6 p.m., one on an early lunch and one on a late lunch; one person begins at 8 p.m. and takes an early lunch; two persons begin at 10 p.m., one on an early lunch and one on a late lunch. The cost of this optimal solution is \$9,084.

It so happens that this solution employs a total 37 operators, as in the original model where costs are not considered. However, some of them are re-allocated in order to minimize the total labour cost. (Clearly, the previous model has multiple *integer* optimal solutions).

3.3 Production Planning Models

When the demand for a product fluctuates over time, the amount produced over time can fluctuate in tandem with the demand, or it may be smooth and instead an inventory is built up in periods of low demand and drawn down in periods of high demand. Between the extreme policies of exactly matching production with demand on the one hand, or a constant rate of production on the other, lie a multiplicity of intermediate policies. The first extreme is most closely attained when a product cannot be kept in inventory – for example, hot dogs at a baseball stadium. The second extreme is most closely attained when the cost of changing the level of production is very high – for example, the smelting of aluminium. In this section we will develop some linear models of production and inventory levels.

3.3.1 A Simple Inventory Model

Tools R Us makes a precision tool for the oil industry. Standard practice in this specialized tool market is to place orders six months in advance of the desired delivery. Hence *Tools R Us* knows that the demand over the next six months will be:

Sep.	Oct.	Nov.	Dec.	Jan.	Feb.
740	800	280	470	630	510

It is now August 1. Based on what has already been planned for this month, the inventory of this tool as of August 31 will be 300 units. Up to 600 units can be manufactured each month based on each employee working on regular time. Each unit costs \$67 if manufactured on regular time. It is also possible to manufacture up to 150 units on overtime. Each unit so produced costs \$95.

Tools can be kept in inventory at a cost of \$2 per unit per month. This charge represents the cost of tied-up capital, warehouse, and insurance expenses. As a buffer against potentially high demand at the end of the planning horizon, the ending inventory should be at least 200 units.

With the restriction that all customers' orders must be completed on time, we wish to formulate a model which seeks to minimize the sum of production and inventory costs.

3.3.2 Formulation

Since this problem deals with months, we will index the months so that we can refer to the index rather than the name of the calendar month. We will use the index t, where $t = 1$ means September and $t = 6$ means February. In problems such as this the initial conditions are important: the month of August is denoted as $t = 0$.

For each month we must decide how many units are to be produced on regular time, and how many are to be produced on overtime. Rather than use the symbolic name X, we will name the variables in a manner which helps us to recall what they represent. We define:

$R_t =$ the number of tools produced on regular time in month t, where $t = 1, \ldots, 6$.

$O_t =$ the number of tools produced on overtime in month t, where $t = 1, \ldots, 6$.

The inventory must be known each month. It is important to distinguish how the inventory is measured, be it the beginning, ending, average, or other measure. We define

$I_t =$ the number of units in inventory at the end of month t, where $t = 1, \ldots, 6$. Although it is not, strictly speaking, a variable, it is useful to think of the inventory at the end of August being represented as I_0.

The objective function is:

$$\text{minimize} \sum_{t=1}^{6}(67R_t + 95O_t + 2I_t)$$

The constraints are of two types. The first and easier type are the capacity constraints; there are twelve of these, six for regular time and six for overtime.

$$R_t \le 600 \qquad (t = 1, \ldots, 6)$$

$$O_t \le 150 \qquad (t = 1, \ldots, 6)$$

The second type of constraint exists, in part, to satisfy the customer requirements. In each month, the initial inventory plus the amount produced during the month must meet or exceed the customer requirements for that month. Hence in the first month we have

$$I_0 + R_1 + O_1 \ge 740$$

Following this argument, in the second month we have

$$I_1 + R_2 + O_2 \ge 800$$

However, I_1 cannot simply take on any value but instead must represent the difference between the left hand side and the right hand side of the previous constraint. In other words, in month 1 the variables must balance in an equality constraint of the form:

$$I_0 + R_1 + O_1 = 740 + I_1$$

Putting this into the standard form where all variables appear on the left we obtain:

$$I_0 + R_1 + O_1 - I_1 = 740 \tag{3.1}$$

Such a constraint is commonly called an *inventory balance constraint*. Since I_0 appears in this constraint, we also need the constraint

$$I_0 = 300$$

Alternatively, the 300 units of initial inventory can be imbedded in the month 1 inventory balance constraint to yield

$$R_1 + O_1 - I_1 = 440$$

Doing it this way, however, obscures the original data on initial inventory and September demand, which weakens the benefits of sensitivity analysis. Therefore, we prefer the format of Equation 3.1.

Similarly, the other five inventory balance constraints are:

$$\begin{aligned}
I_1 + R_2 + O_2 - I_2 &= 800 \\
I_2 + R_3 + O_3 - I_3 &= 280 \\
I_3 + R_4 + O_4 - I_4 &= 470 \\
I_4 + R_5 + O_5 - I_5 &= 630 \\
I_5 + R_6 + O_6 - I_6 &= 510
\end{aligned}$$

Finally, we require that $I_6 \geq 200$, and all variables must be non-negative. The complete formulation has nineteen variables and twenty constraints (Note: each of the first two rows after "subject to" uses one line to define *six* constraints):

$$\begin{array}{rrcll}
\text{minimize} & \displaystyle\sum_{t=1}^{6} (67R_t + 95O_t + 2I_t) & & & \\
\text{subject to} & & & & \\
\text{Regular time production} & R_t & \leq & 600 & (t = 1, \ldots, 6) \\
\text{Overtime production} & O_t & \leq & 150 & (t = 1, \ldots, 6) \\
\text{Initial inventory} & I_0 & = & 300 & \\
\text{Inventory balance, month 1} & I_0 + R_1 + O_1 - I_1 & = & 740 & \\
\text{Inventory balance, month 2} & I_1 + R_2 + O_2 - I_2 & = & 800 & \\
\text{Inventory balance, month 3} & I_2 + R_3 + O_3 - I_3 & = & 280 & \\
\text{Inventory balance, month 4} & I_3 + R_4 + O_4 - I_4 & = & 470 & \\
\text{Inventory balance, month 5} & I_4 + R_5 + O_5 - I_5 & = & 630 & \\
\text{Inventory balance, month 6} & I_5 + R_6 + O_6 - I_6 & = & 510 & \\
\text{Ending inventory} & I_6 & \geq & 200 & \\
 & \text{all variables must be} \geq 0 & & &
\end{array}$$

An OFV of \$225,470 was found. The values of the variables are:

t	1	2	3	4	5	6
R_t	600	600	290	600	600	600
O_t	0	40	0	0	0	0
I_t	160	0	10	140	110	200

Given that overtime is much more expensive than regular time, overtime is used only when necessary. In the second month, 40 units are produced on overtime in order to meet the balance of that month's requirement. Month three's demand is relatively low, so under the assumptions of this model the production is cut accordingly.

To this basic model we now consider separately two extensions. In one, the demand can be 'back-ordered' at a certain cost. In the other, changes to the production level from month to month have a cost associated with them.

3.3.3 Shortage Model

Companies cannot always deliver their products when they are demanded. Whenever this happens, however, they risk losing the customer's future business. There may be other costs as well, for example the cost of expediting a shipment using a courier service. In modelling the possibility of late deliveries, it is customary to assign a high penalty cost for lateness. Suppose that *Tools R Us* gives its customers a \$25 per tool rebate for each month that the product is delivered late. They could then take this figure as the cost of a late product and add this possibility to the basic model. When late deliveries are permitted, it is possible for the inventory to be negative, meaning that the customers' requirements for that month have been partially back-ordered.

It may appear that we have found a variable for which the non-negativity restriction does not apply. However, within the formulated model, we will not let the inventory variable wander into the negative region, because this gains us nothing. Instead, we will model the inventory level in each month using two variables, one to represent inventory-on-hand, and the other to represent back-ordered inventory. Each of these two variables must be non-negative, and the context requires that in each month at least one of these two variables must be zero. The amount of inventory-on-hand at the end of month t, where $t = 0, \ldots, 6$, will be denoted as I_t^+, and the amount of back-ordered inventory at the end of month t is denoted as I_t^-. The net inventory level in month t, denoted as I_t, can be positive or negative.

These three variables are related by the equation

$$I_t = I_t^+ - I_t^-$$

In the objective function, each $2I_t$ is replaced by $2I_t^+$, and we add $\sum_{t=1}^{6} 25I_t^-$. In this model there is no need to force at least one of I_t^+ or I_t^- to be zero – the objective function as it stands will ensure that this logical requirement is met.

In each of the constraints, for the appropriate value of t, I_t is replaced by $I_t^+ - I_t^-$ wherever it appears. Making these changes we obtain a model with twenty-six variables and twenty constraints:

$$\begin{array}{rrcll}
\text{minimize} & \sum_{t=1}^{6}(67R_t + 95O_t + 2I_t^+ + 25I_t^-) & & & \\
\text{subject to} & & & & \\
\text{Regular time production} & R_t & \leq & 600 & (t = 1, \ldots, 6) \\
\text{Overtime production} & O_t & \leq & 150 & (t = 1, \ldots, 6) \\
\text{Initial inventory} & I_0^+ - I_0^- & = & 300 & \\
\text{Inventory balance, month 1} & I_0^+ - I_0^- + R_1 + O_1 - I_1^+ + I_1^- & = & 740 & \\
\text{Inventory balance, month 2} & I_1^+ - I_1^- + R_2 + O_2 - I_2^+ + I_2^- & = & 800 & \\
\text{Inventory balance, month 3} & I_2^+ - I_2^- + R_3 + O_3 - I_3^+ + I_3^- & = & 280 & \\
\text{Inventory balance, month 4} & I_3^+ - I_3^- + R_4 + O_4 - I_4^+ + I_4^- & = & 470 & \\
\text{Inventory balance, month 5} & I_4^+ - I_4^- + R_5 + O_5 - I_5^+ + I_5^- & = & 630 & \\
\text{Inventory balance, month 6} & I_5^+ - I_5^- + R_6 + O_6 - I_6^+ + I_6^- & = & 510 & \\
\text{Ending inventory} & I_6^+ - I_6^- & \geq & 200 & \\
 & \text{all variables must be} & \geq & 0 &
\end{array}$$

An OFV of \$225,350 was found, a savings of \$120 over the basic model. The optimal values of the variables are:

t	1	2	3	4	5	6
R_t	600	600	330	600	600	600
O_t	0	0	0	0	0	0
I_t^+	160	0	10	140	110	200
I_t^-	0	40	0	0	0	0

The overtime in the previous model (40 units in month 2) has been transferred to month 3's regular time production, saving 40 times \$95 in overtime charges, but adding 40 times \$67 in regular time charges plus 40 times \$25 in rebate charges, for a net savings of 40(\$95 − \$67 − \$25) = \$120.

3.3.4 Production Level Change Model

It is often the case that changes in the production level, either upward or downward, incur costs. When the production level is increased, there may be costs in hiring new workers, training these workers, and adjusting the production machinery. When the production level is decreased, there may be a legal requirement to give severance pay. We now present a model which considers changes to the production level. To keep the model as simple as possible, shortages will not be allowed.

In such a model we need to know the initial and final conditions for the *production level* as well as these conditions for the inventory level. Suppose that the data are as they were in the original model, but we now add the initial condition that the production level for August is 570 units (all regular time), and we add the final condition that the desired production level for February is between 250 and 500 units. The cost to increase the production level from one month to the next is \$17 per unit increased, and the cost to decrease the production level from one month to the next is \$38 per unit decreased.

In addition to the R_t, O_t, and I_t variables, we need variables to represent the changes in the production level from one month to the next. We define:

$U_t =$ the increase in the production level from month $t-1$ to month t, where $t = 1, \dots, 6$.

$D_t =$ the decrease in the production level from month $t-1$ to month t, where $t = 1, \dots, 6$.

Adding these variables to the objective function we obtain:

$$\text{minimize} \sum_{t=1}^{6} (67R_t + 95O_t + 2I_t + 17U_t + 38D_t)$$

To the constraints of the original model we add the following. First, there are the initial and final conditions:

$$\begin{aligned} R_0 &= 570 \\ O_0 &= 0 \\ R_6 + O_6 &\geq 250 \\ R_6 + O_6 &\leq 500 \end{aligned}$$

Secondly, there is a set of constraints which relate the production level change variables to the production level variables. The increase or decrease in the production level from August to September is given by:

$$R_1 + O_1 - R_0 - O_0$$

This expression, be it positive or negative, must equal $U_1 - D_1$. Hence

$$R_1 + O_1 - R_0 - O_0 = U_1 - D_1$$

Therefore:

$$R_1 + O_1 - R_0 - O_0 - U_1 + D_1 = 0$$

This is the *production level change balance constraint* for month 1. For the other five months we have:

$$\begin{aligned}
R_2 + O_2 - R_1 - O_1 - U_2 + D_2 &= 0 \\
R_3 + O_3 - R_2 - O_2 - U_3 + D_3 &= 0 \\
R_4 + O_4 - R_3 - O_3 - U_4 + D_4 &= 0 \\
R_5 + O_5 - R_4 - O_4 - U_5 + D_5 &= 0 \\
R_6 + O_6 - R_5 - O_5 - U_6 + D_6 &= 0
\end{aligned}$$

This production level change model has thirty-three variables and thirty constraints.

$$
\begin{array}{rrcll}
\text{minimize} & \displaystyle\sum_{t=1}^{6}(67R_t+95O_t+2I_t+17U_t+38D_t) & & & \\
\text{subject to} & & & & \\
\text{Regular time production} & R_t & \leq & 600 & (t=1,\ldots,6) \\
\text{Overtime production} & O_t & \leq & 150 & (t=1,\ldots,6) \\
\text{Initial inventory} & I_0 & = & 300 & \\
\text{Inventory balance, month 1} & I_0+R_1+O_1-I_1 & = & 740 & \\
\text{Inventory balance, month 2} & I_1+R_2+O_2-I_2 & = & 800 & \\
\text{Inventory balance, month 3} & I_2+R_3+O_3-I_3 & = & 280 & \\
\text{Inventory balance, month 4} & I_3+R_4+O_4-I_4 & = & 470 & \\
\text{Inventory balance, month 5} & I_4+R_5+O_5-I_5 & = & 630 & \\
\text{Inventory balance, month 6} & I_5+R_6+O_6-I_6 & = & 510 & \\
\text{Ending inventory} & I_6 & \geq & 200 & \\
\text{Initial regular time production} & R_0 & = & 570 & \\
\text{Initial overtime production} & O_0 & = & 0 & \\
\text{Min. total ending production} & R_6+O_6 & \geq & 250 & \\
\text{Max. total ending production} & R_6+O_6 & \leq & 500 & \\
\text{Production change balance, month 1} & R_1+O_1-R_0-O_0-U_1+D_1 & = & 0 & \\
\text{Production change balance, month 2} & R_2+O_2-R_1-O_1-U_2+D_2 & = & 0 & \\
\text{Production change balance, month 3} & R_3+O_3-R_2-O_2-U_3+D_3 & = & 0 & \\
\text{Production change balance, month 4} & R_4+O_4-R_3-O_3-U_4+D_4 & = & 0 & \\
\text{Production change balance, month 5} & R_5+O_5-R_4-O_4-U_5+D_5 & = & 0 & \\
\text{Production change balance, month 6} & R_6+O_6-R_5-O_5-U_6+D_6 & = & 0 & \\
 & & & & \\
 & \text{all variables must be} & \geq & 0 &
\end{array}
$$

For clarity, the last six constraints have been shown in full, but we could simplify this set of constraints to:

$$
\begin{array}{l}
\text{Production change balance, month } t\text{:} \\
\qquad R_t+O_t-R_{t-1}-O_{t-1}-U_t+D_t \quad = \quad 0 \qquad (t=1,\ldots,6)
\end{array}
$$

An OFV of \$231,940 was found. This is a higher cost than that of the basic model, since the changes added only new costs without any improvements in flexibility. The optimal values of the variables are:

t	1	2	3	4	5	6
R_t	600	600	530	530	530	500
O_t	20	20	0	0	0	0
I_t	180	0	250	310	210	200
U_t	50	0	0	0	0	0
D_t	0	0	90	0	0	30

At the outset, the production level is increased in order to meet the shipping requirements of the first two months. Thereafter, the production level falls, first from month 2 to month 3, and then from month 5 to month 6. The second decline was required because of the constraint on total production in month 6. This type of model smooths the production level, thereby reducing the need for layoffs and re-hiring.

3.4 Cutting Stock Models

A common industrial problem is that of determining how to cut stock in order to meet a customer's specific requirements. This *cutting stock* problem could involve such products as rolls of paper or aluminium foil; the examples considered here are concerned with wooden dowels. The first example highlights the concepts, while the second is a more challenging problem.

3.4.1 Example 1 – Description

A lumber yard stocks 1 cm diameter wooden dowels in a standard width of 1 metre. A customer comes into the yard seeking seven dowels of width 45 cm, thirteen of width 37 cm, and eight dowels of width 24 cm. They wish to cut the customer's order so as to minimize the number of standard sized dowels used.

If they cut one 45 cm dowel from a one metre (100 cm) dowel, what should they do with the 55 cm dowel which is left over? They could cut a second 45 cm dowel, leaving $100-2(45)=10$ cm of waste. Another option would be to cut off 37 cm from the 55 cm, leaving 18 cm of waste. A third option would be to cut two 24 cm dowels, leaving 7 cm of waste. Each of these is a pattern. Patterns 1, 2, and 3 cut, respectively: two 45's; one 45 and one 37; and one 45 and two 24's.

We can also examine what happens when no 45 is cut. We can divide 37 into 100 twice with a remainder, hence we could cut two 37's, and with the other

Pattern	45 cm	37 cm	24 cm	Waste (cm)
1	2	0	0	10
2	1	1	0	18
3	1	0	2	7
4	0	2	1	2
5	0	1	2	15
6	0	0	4	4

Table 3.1: Example 1 – List of the Patterns.

$100-2(37)=26$ cm, cut one 24 cm dowel from it. Hence Pattern 4 cuts no 45, two 37's, and one 24. For Pattern 5 we could cut one 37 cm dowel, leaving 63 cm, from which we could cut two 24's, with $63-2(24)=15$ cm of waste. Finally, for Pattern 6 if we cut no 45's and no 37's, we would be able to cut four 24's from the 100 cm standard dowel, leaving 4 cm of waste. A list of all the patterns is shown in Table 3.1.

Let X be the number of standard size dowels used, and let P_i represent the number of patterns of type i cut, where $i=1,\ldots,6$. The objective is simply to minimize X. Since X will be the sum of the P_i's, there will be a balance constraint:

$$-X+P_1+P_2+P_3+P_4+P_5+P_6=0$$

or

$$-X+\sum_{i=1}^{6}P_i=0$$

Alternatively, we could omit X and the balance constraint, and simply minimize $P_1+P_2+P_3+P_4+P_5+P_6$. The number of standard dowels used is simply the OFV.

There will be three other constraints to make sure that the customer's order is met. First, we require that there be at least seven dowels of width 45 cm. Pattern 1 makes two 45's, and Patterns 2 and 3 make one each, hence we require that:

$$2P_1+P_2+P_3\geq 7$$

Secondly, we need thirteen dowels of width 37 cm. These are produced by patterns 2, 4, and 5, with Pattern 4 producing two 37's and the others one each. Hence we require that:

$$P_2+2P_4+P_5\geq 13$$

Two 24's are made by Pattern 3, one by Pattern 4, two by Pattern 5, and four by Pattern 6. We need eight 24's, so the constraint is:

$$2P_3 + P_4 + 2P_5 + 4P_6 \geq 8$$

Finally, we note that we must produce an integer number of patterns.

From the list of the patterns in Table 3.1, it is easy to obtain the left-hand side coefficients for the last three constraints, which come from the middle three columns. For example, in the 45 cm column, the numbers are 2, 1, 1, 0, 0, 0 creating

$$2P_1 + 1P_2 + 1P_3 + 0P_4 + 0P_5 + 0P_6 = 2P_1 + P_2 + P_3$$

Another way of looking at this is that the numbers on the left-hand side of the 45 cm row in the Excel model will be 2, 1, 1, 0, 0, and 0.

The model is:

$$\begin{array}{rrcl}
\text{minimize} & X & & \\
\text{subject to} & & & \\
\text{Balance} & -X + \sum_{i=1}^{6} P_i & = & 0 \\
\text{45 cm} & 2P_1 + P_2 + P_3 & \geq & 7 \\
\text{37 cm} & P_2 + 2P_4 + P_5 & \geq & 13 \\
\text{24 cm} & 2P_3 + P_4 + 2P_5 + 4P_6 & \geq & 8
\end{array}$$

$$\text{all variables } \geq 0 \text{ and integer}$$

An alternate model is:

$$\begin{array}{rrcl}
\text{minimize} & \sum_{i=1}^{6} P_i & & \\
\text{subject to} & & & \\
\text{45 cm} & 2P_1 + P_2 + P_3 & \geq & 7 \\
\text{37 cm} & P_2 + 2P_4 + P_5 & \geq & 13 \\
\text{24 cm} & 2P_3 + P_4 + 2P_5 + 4P_6 & \geq & 8
\end{array}$$

$$\text{all variables } \geq 0 \text{ and integer}$$

Entering the latter model into Excel we obtain:

	A	B	C	D	E	F	G	H	I	J
1										
2	OFV	**Cutting Stock Model**								
3	0	P1	P2	P3	P4	P5	P6			
4	Minimize	1	1	1	1	1	1			
5	# of Patterns									
6										
7	Dowels									RHS
8	45 cm	2	1	1				0	>=	7
9	37 cm		1		2	1		0	>=	13
10	24 cm			2	1	2	4	0	>=	8

In this example, just as in the constable staffing problem seen earlier, we could have omitted the 1's in row 4, and simply have entered =SUM(B3:I3) into cell A3, but for consistency with the other Excel models we kept the 1's and used the **SUMPRODUCT** function. In addition to checking the box to declare non-negativity, we use the *Add Constraint* dialog box, declaring the range `B5:G5` to be "int" (middle box).

	A	B	C	D	E	F	G	H	I	J
1										
2	OFV	**Cutting Stock Model**								
3	11	P1	P2	P3	P4	P5	P6			
4	Minimize	1	1	1	1	1	1			
5	# of Patterns	3	1	0	5	2	0			
6										
7	Dowels									RHS
8	45 cm	2	1	1				7	>=	7
9	37 cm		1		2	1		13	>=	13
10	24 cm			2	1	2	4	9	>=	8

The optimal solution is to use 11 standard size dowels to make three pattern 1's, one pattern 2, five pattern 4's, and two pattern 5's.

3.4.2 Example 2 (Optional)

Description and Formulation

A mill produces dowels in standard lengths of 80, 150, and 200 cm. All dowels have a diameter of 1 cm. These lengths are then sold and shipped wholesale to lumber yards. A retail customer needs ten 26 cm dowels, fourteen 73 cm dowels, and twenty 118 cm dowels. The lumber yard wishes to cut what the customer wants using standard length dowels. For example, two 73 cm dowels could be cut from one 150 cm dowel, the other 4 cm being waste. We wish to formulate a model which will show how to cut the custom-made dowels so that the amount of wasted wood is minimized.

It is possible to directly minimize the amount of wasted wood, but it is easier to do so indirectly. This is accomplished by seeing that since the total length of dowel required by the customer is fixed (3642 cm), the amount of wasted wood is minimized by minimizing the total length of standard dowels used. Hence we define:

$$\begin{aligned} X_1 &= \text{the number of 80 cm dowels used} \\ X_2 &= \text{the number of 150 cm dowels used} \\ X_3 &= \text{the number of 200 cm dowels used} \end{aligned}$$

The total length of standard sized dowels used is $80X_1 + 150X_2 + 200X_3$. Hence the objective function is

$$\text{minimize} \quad 80X_1 + 150X_2 + 200X_3$$

What complicates this formulation is that it is not obvious how each standard sized dowel should be used. For example, each 80 cm dowel could be used to cut one 73 cm dowel, with 7 cm of waste, or three 26 cm dowels, with 2 cm of waste.

As we saw from the previous example, these are patterns, but the difference is that we must keep track of the standard-size dowels used to make these patterns. In examples such as this, each pattern has a label of the form (i, j), where i represents a standard-size dowel and j is a pattern cut from it. Let the two patterns cut from the 80 cm dowels be patterns (1,1) and (1,2) respectively. In terms of the customer's requirements for 118, 73, and 26 cm dowels[8] pattern (1,1) cuts [0,1,0] and pattern (1,2) cuts [0,0,3]. Either or both of these patterns could appear in the optimal solution. We will disregard any pattern which is trivially sub-optimal,

[8] It is easier to consider the custom-made dowels in *descending* order of length.

such as cutting two 26 cm dowels from one 80 cm dowel, resulting in 28 cm of waste.[9] Hence the patterns of interest are:

Patterns from an 80 cm Dowel for 118, 73, and 26 cm Dowels

Label	Pattern Cuts	Waste
(1,1)	[0,1,0]	7 cm
(1,2)	[0,0,3]	2 cm

A 150 cm dowel can be used to cut one 118 cm dowel, leaving 32 cm from which one 26 cm dowel can be cut. Hence pattern (2,1) cuts [1,0,1] with 6 cm of waste. If no 118 cm dowels are cut, then we can cut up to two 73 cm dowels, leaving 4 cm of waste; pattern (2,2) cuts [0,2,0]. Cutting no 118 cm dowel and just one 73 cm dowel leaves 77 cm, from which two 26 cm dowels can be cut, leaving 25 cm of waste. Obviously, the amount of waste from any pattern must be less than the length of the shortest dowel (26 cm). Pattern (2,3) which cuts [0,1,2] creating 25 cm of waste is not likely to be in the optimal solution, but since the waste is less than 26 cm, it cannot be ruled out at this time. Finally, if no 118 or 73 cm dowels are cut then we have pattern (2,4) which cuts [0,0,5], with 20 cm of waste. Hence the patterns of interest from a 150 cm dowel are:

Patterns from a 150 cm Dowel for 118, 73, and 26 cm Dowels

Label	Pattern Cuts	Waste
(2,1)	[1,0,1]	6 cm
(2,2)	[0,2,0]	4 cm
(2,3)	[0,1,2]	25 cm
(2,4)	[0,0,5]	20 cm

Repeating this procedure for the 200 cm dowels the patterns are:

Patterns from a 200 cm Dowel for 118, 73, and 26 cm Dowels

Label	Pattern Cuts	Waste
(3,1)	[1,1,0]	9 cm
(3,2)	[1,0,3]	4 cm
(3,3)	[0,2,2]	2 cm
(3,4)	[0,1,4]	23 cm
(3,5)	[0,0,7]	18 cm

[9] We can think of this as [0,0,3] *dominating* [0,0,2].

Now we need to add some more variables. We define

$$P_{ij} = \text{ the number of patterns of type } (i,j) \text{ used}$$

where i and j range as given above.

We need to balance the number of standard sized dowels of size i used with the number of patterns of type (i,j) cut. This gives the following three equations:

$$\begin{aligned} P_{1,1}+P_{1,2}-X_1 &= 0 \\ P_{2,1}+P_{2,2}+P_{2,3}+P_{2,4}-X_2 &= 0 \\ P_{3,1}+P_{3,2}+P_{3,3}+P_{3,4}+P_{3,5}-X_3 &= 0 \end{aligned}$$

We need at least twenty 118 cm dowels. This length is produced by three patterns, each of which produces one 118 cm dowel. Hence

$$P_{2,1}+P_{3,1}+P_{3,2} \geq 20$$

We must produce at least fourteen 73 cm dowels. There are six patterns which are applicable; four which produce one 73 cm dowel and two which produce two 73 cm dowels.

$$P_{1,1}+2P_{2,2}+P_{2,3}+P_{3,1}+2P_{3,3}+P_{3,4} \geq 14$$

Finally, we need at least ten 26 cm dowels. There are eight patterns which produce one or more 26 cm dowels. Summing over all of these we obtain:

$$3P_{1,2}+P_{2,1}+2P_{2,3}+5P_{2,4}+3P_{3,2}+2P_{3,3}+4P_{3,4}+7P_{3,5} \geq 10$$

This problem, like the telephone operator problem, requires that all variables be $\in \{0,1,2,3,\ldots\}$. The complete formulation is:

$$\begin{array}{rrcl}
\text{minimize} & 80X_1+150X_2+200X_3 & & \\
\text{subject to} & & & \\
\text{Balance, 80 cm standard dowels} & P_{1,1}+P_{1,2}-X_1 & = & 0 \\
\text{Balance, 150 cm standard dowels} & P_{2,1}+P_{2,2}+P_{2,3}+P_{2,4}-X_2 & = & 0 \\
\text{Balance, 200 cm standard dowels} & P_{3,1}+P_{3,2}+P_{3,3}+P_{3,4}+P_{3,5}-X_3 & = & 0 \\
\text{118 cm custom dowels} & P_{2,1}+P_{3,1}+P_{3,2} & \geq & 20 \\
\text{73 cm custom dowels} & P_{1,1}+2P_{2,2}+P_{2,3}+P_{3,1}+2P_{3,3}+P_{3,4} & \geq & 14 \\
\text{26 cm custom dowels} & 3P_{1,2}+P_{2,1}+2P_{2,3}+5P_{2,4}+ & & \\
 & 3P_{3,2}+2P_{3,3}+4P_{3,4}+7P_{3,5} & \geq & 10 \\
 & & & \\
 & \text{all variables } \in \{0,1,2,3,\ldots\} & &
\end{array}$$

We solve this on the Excel Solver, including a declaration that the range of variable cells be "int" (integer). There are multiple optima; one optimal solution is $X_2 = 4$, $X_3 = 16$, $P_{2,1} = 4$, $P_{3,1} = 14$, and $P_{3,2} = 2$, with all other variables being 0. The optimal OFV is $150 \times 4 + 200 \times 16 = 3800$ cm. Since 3642 cm of dowel is sent to the customer, this optimal plan creates 158 cm of waste.

It turns out that the above solution is naturally integer, meaning that if we had omitted the *int* declaration we would have found an integer solution anyway. A slight change can destroy this harmony. If, for example, the customer's requirement for 26 cm dowels had been only 7 instead of 10, the solution is no longer naturally integer. Since one does not know ahead of time if such a model will be naturally integer, it makes sense to make the *int* declaration whenever the variables must be integer.

An Alternate Model (Optional)

Knowing the waste created by each pattern, and the numbers of each pattern cut, the total amount of waste is given by:

$$\begin{aligned}\text{total waste} \;=\; & 7P_{1,1} + 2P_{1,2} + 6P_{2,1} + 4P_{2,2} + 25P_{2,3} + 20P_{2,4} + \\ & 9P_{3,1} + 4P_{3,2} + 2P_{3,3} + 23P_{3,4} + 18P_{3,5}\end{aligned}$$

This expression can form part of an objective function, but we must also penalize any customer lengths which are made in excess of the customer's order. To do this we define S_1, S_2, and S_3 as the number of dowels of length 118, 73, and 26 cm respectively made in excess of the customer's requirements. Subtracting these variables from the left-hand side turns the custom constraints into equalities. These constraints are now

$$\begin{array}{lrcl}\text{118 cm custom dowels} & P_{2,1} + P_{3,1} + P_{3,2} - S_1 & = & 20 \\ \text{73 cm custom dowels} & P_{1,1} + 2P_{2,2} + P_{2,3} + P_{3,1} + 2P_{3,3} + P_{3,4} - S_2 & = & 14 \\ \text{26 cm custom dowels} & 3P_{1,2} + P_{2,1} + 2P_{2,3} + 5P_{2,4} + & & \\ & 3P_{3,2} + 2P_{3,3} + 4P_{3,4} + 7P_{3,5} - S_3 & = & 10\end{array}$$

The dowels which, if any, are produced in excess of those demanded, are penalized by putting the terms

$$118S_1 + 73S_2 + 26S_3$$

into the objective function. The first three constraints, the balance constraints on the standard size dowels, are no longer required.[10] The alternate model is therefore:

[10] However, there would be no harm in leaving them in if had wanted to do so.

$$\begin{array}{rrcl}
\text{minimize} & 7P_{1,1}+2P_{1,2}+6P_{2,1}+4P_{2,2}+25P_{2,3}+20P_{2,4}+ & & \\
 & 9P_{3,1}+4P_{3,2}+2P_{3,3}+23P_{3,4}+18P_{3,5} & & \\
 & +118S_1+73S_2+26S_3 & & \\
\text{subject to} & & & \\
\text{118 cm custom dowels} & P_{2,1}+P_{3,1}+P_{3,2}-S_1 & = & 20 \\
\text{73 cm custom dowels} & P_{1,1}+2P_{2,2}+P_{2,3}+P_{3,1}+2P_{3,3}+P_{3,4}-S_2 & = & 14 \\
\text{26 cm custom dowels} & 3P_{1,2}+P_{2,1}+2P_{2,3}+5P_{2,4}+ & & \\
 & 3P_{3,2}+2P_{3,3}+4P_{3,4}+7P_{3,5}-S_3 & = & 10 \\
 & & & \\
 & \text{all variables } \in \{0,1,2,3,\ldots\} & &
\end{array}$$

This model will have a different OFV*; it will differ from the previous one by the fixed length requirement of 3642 cm. All X_i's and P^*_{ij} values will be the same as before. Note that each of the objective function coefficients here is a calculated value, which can be a source of error. Therefore, the original objective function written in terms of the X_i's is the preferred form.

3.5 Some Special Situations

In this section we consider some situations which lead to potential problems in their formulation.

Ratio Constraints

We have seen some of these already, but sometimes the wording might trick us. Consider, for example, a company which makes tables and chairs, and in the problem description it is stated that "for every table made, they must make at least four chairs". If T and C represent respectively the number of tables and chairs made, a common mistake is to write this as $T \geq 4C$. This happens because the T, the 4, and the C, all appear in this order in the sentence. However we know that if they make 20 tables, then they must make at least 80 chairs. Hence it is T (rather than C) which must be multiplied by 4, i.e. $C \geq 4T$, which in standard form is:

$$\text{Ratio} \qquad -4T + C \geq 0$$

A good idea in these situations is to make a numerical example, and then test the constraint to see that things are working properly.

Confusing a Coefficient with its Reciprocal

Suppose that a machine is used to make two models of circuit boards. Type 1 boards can be made at a rate of four per hour, and Type 2 boards can be made at a rate of five per hour. The machine is available for ten hours per day. Let X_1 and X_2 represent respectively the number of Type 1 and Type 2 circuit boards made each day. A common mistake is to write the daily production constraint as $4X_1 + 5X_2 \leq 10$. This comes from confusing "four per hour" with four hours to make one board. Since we can make four per hour, it only takes one-quarter of a hour to make one Type 1 circuit board. Hence the proper way to write this constraint is:

$$\text{Machine Availability} \qquad \frac{X_1}{4} + \frac{X_2}{5} \leq 10$$

Writing the constraint this way preserves the original data of the problem. This information is lost if we convert it to the decimal form $0.25X_1 + 0.2X_2 \leq 10$. While some optimization software would require the decimal form, in Excel we can simply leave the constraint as it is and enter the data into Excel as `=1/4` and `=1/5`. Certainly, if the denominators had been numbers like 7 and 11, it would be best just to leave the constraint as it is.

Shared Time

Suppose that a crusher can crush Type 1 rock at a rate of 800 Tonnes per hour, but the much harder Type 2 rock can be crushed at a rate of only 400 Tonnes per hour. Let X_1 and X_2 represent respectively the number of Tonnes of Type 1 and Type 2 rock crushed each hour. A common mistake is to model this with two constraints $X_1 \leq 800$ and $X_2 \leq 400$. While these must be true, they do not capture the sharing of time on the crusher.

The way to handle this situation is to imagine one hour of the crusher's time. During this hour, some of the time could be crushing Type 1 rock, some could be crushing Type 2 rock, and some could be idle. The fraction of the hour spent crushing Type 1 rock is $X_1/800$, and the fraction of the hour spent crushing Type 1 rock is $X_1/400$. The sum of these fractions cannot exceed 1, hence:

$$\text{Crusher} \qquad \frac{X_1}{800} + \frac{X_2}{400} \leq 1$$

The difference between the right-hand side and the numerical value of the left-hand side is the fraction of the hour in which the crusher is idle.

Notice that the format of this constraint has much in common with the machine availability constraint in the previous example. Similarly, it can be expressed differently, such as $0.00125X_1 + 0.0025X_2 \leq 1$, or $X_1 + 2X_2 \leq 800$, but the first way preserves the original data of the problem.

Buying Extra Resources

Suppose that a farmer has 80,000 cubic metres of water available from collected rainfall. A consultant has made an algebraic model for the farmer in which the variables represent the number of hectares of land devoted to three types of crops. Based on this, the consultant has come up with a water availability constraint of:

$$\text{Water Availability} \qquad 20X_1 + 90X_2 + 75X_3 \leq 80,000$$

Now suppose that things are as they were before, except that if desired the farmer can buy up to an extra 60,000 cubic metres of water from an irrigation system at a cost of five cents per cubic metre. This one sentence will require that four things be done:

1. We need to define a variable for the amount in cubic metres of extra water bought. Suppose that we name this variable W.

2. In a profit maximization problem, the water is a cost, so its coefficient will be negative. The objective function in dollars will be as it was before, but with a new term of $-0.05W$ added to it.

3. The purchased water adds to the water already available, hence:

   $$\text{Water Availability} \qquad 20X_1 + 90X_2 + 75X_3 \leq 80,000 + W$$

   In standard form this is:

   $$\text{Water Availability} \qquad 20X_1 + 90X_2 + 75X_3 - W \leq 80,000$$

4. Finally, we need an extra constraint:

   $$\text{Water Purchased} \qquad W \leq 60,000$$

3.6 Summary

This chapter has illustrated examples from four applications of linear optimization: blending, scheduling, production planning, and the cutting stock problem. In all cases we needed to examine the unknowns of the situation, which are represented by decision variables. The objective gives us an idea of what some of these variables are; others are more subtle and we may not discover them until we try to write the constraints.

3.7 Problems for Student Completion

Formulate the following problems as linear optimization models. Where appropriate, you may wish to number the commodities and then use subscripted variables. In some cases no specific objective is named; it is up to the reader to come up with an appropriate objective.

3.7.1 Blending Gasoline

A company blends two gasolines from Avalon Fuels and Bonavista Petrol (inputs) into two commercial products, Extra and Regular gasoline (outputs). For the inputs, the octane ratings, the vapour pressures in kilopascals, and the amounts available in cubic metres (m^3) and their prices are known. These are:

Input Gasoline	Octane Rating	Vapour Pressure (kPa)	Amount Available (m^3)	Buying Price ($ per m^3)
Avalon	99	38	55,000	520
Bonavista	82	55	80,000	440

For the Extra and Regular gasolines the requirements are:

Output Gasoline	Minimum Octane Rating	Maximum Vapour Pressure (kPa)	Minimum Amount Required (m^3)	Selling Price ($ per m^3)
Extra	94	40	36,000	540
Regular	86	52	70,000	470

(a) Make an algebraic model for this problem, where the variables are defined as follows: E and R are respectively the amount of Extra/Regular gasoline in m^3 blended and sold; A and B are respectively the amount of gasoline in m^3 purchased from Avalon Fuels/Bonavista Petrol; AE, AR, BE, and BR are respectively the amounts in m^3 of Avalon/Bonavista gasoline used to make Extra/Regular gasoline.

(b) Make a spreadsheet model, and solve it using the Excel Solver.

(c) State the recommendation clearly.

3.7.2 Blending Oil

An oil refinery has three types of inputs, with the following prices and characteristics:

Input #	Price per litre	% Sulphur (by mass)	Thermal Value (kilojoules/litre)
1	\$0.42	2.2	15,000
2	\$0.76	0.4	20,000
3	\$0.60	1.0	17,000

The inputs are blended to produce two outputs, with the following outputs and promised specifications:

Output #	Price per litre	Maximum % Sulphur (by mass)	Minimum Thermal Value (kilojoules/litre)
1	\$0.63	1.2	16,000
2	\$0.91	0.7	18,000

The refinery has a capacity of 1,000,000 litres/day overall. Subject to the overall capacity, up to 500,000 litres/day of any input, or 650,000 litres/day of any output can be handled.

We can assume that all three inputs have identical densities, thereby enabling the sulphur percentages to be treated as if they were by volume. We can also assume that there are no losses in the blending process, and that the characteristics of the outputs are a weighted average (by volume) of the characteristics of the inputs.

3.7.3 Scheduling of Restaurant Workers

A large unionized restaurant is planning its workforce schedule. The requirements for employees over the seven day work week are:

Day	Minimum Number of Employees
Sunday	110
Monday	81
Tuesday	85
Wednesday	118
Thursday	124
Friday	112
Saturday	120

The collective bargaining agreement states that all employees are to work five consecutive days per week with two consecutive days off (Saturday and Sunday are consecutive). Such a schedule might mean that some employees show up for work (for which they are paid) but they are not required (for example, the schedule might assign 119 employees on Wednesday). The restaurant manager wishes to minimize such wastage.

3.7.4 An Irrigation Problem

A farmer owns 500 hectares of land in an arid region. The state government gives him up to 1,000,000 cubic metres of water for irrigation each year. In addition, he may purchase up to an additional 300,000 cubic metres of water per annum at a cost of \$0.04 per cubic metre.

He grows corn, peas, and onions. The net revenue per hectare of each commodity (excluding the cost of purchased water, if any) and the water requirement in cubic metres per hectare are:

Commodity	Revenue per Hectare	Water Requirement (cubic metres per hectare)
Corn	\$200	3500
Peas	\$400	6700
Onions	\$300	2000

He wishes to diversify his crop in case one commodity suffers an unanticipated fall in price. Therefore, no commodity may occupy more than 50% of the total area planted, nor may any commodity occupy less than 10% of the total area planted.

3.7.5 Blending of Coffee

Columbian, Peruvian, and Nigerian coffee beans can be purchased for \$1.20, \$1.00, and \$0.90 per kilogram respectively. From these sources a company makes a "regular" and a "premium" blend of coffee, which sell for \$1.30 and \$1.60 per kilogram respectively. The regular blend contains at least 10% (by mass) Columbian beans, and at least 20% Peruvian beans. The premium blend contains at least 50% Columbian beans, and no more than 15% Nigerian beans. The maximum market demand is for 200,000 kilograms of regular coffee and for 130,000 kilograms of premium coffee.

3.7.6 Scheduling of Bus Drivers

City bus drivers work two three and a half-hour shifts per day. In some cases, the two shifts are consecutive (effectively one seven-hour shift), but usually they are not. Because of the inconvenience of breaking up the day, those who work non-consecutive shifts are paid a \$15 per day bonus. All bus drivers earn a base rate of \$25 per hour. The bus company has the following daily requirements:

Time of Day	Minimum Number of Drivers Needed
5:30 a.m. to 8:59 a.m.	150
9:00 a.m. to 12:29 p.m.	80
12:30 p.m. to 3:59 p.m.	90
4:00 p.m. to 7:29 p.m.	160
7:30 p.m. to 10:59 p.m.	70

Subject to meeting all its requirements for drivers, the bus company wishes to minimize its daily labour cost (regular and bonus).

3.7.7 Production Planning 1

A manufacturer of school buses has firm orders for the next four quarters:

Quarter	1	2	3	4
Demand	350	400	290	380

It is now the end of the year. Based on what has already been planned for this quarter, the inventory of school buses as of December 31 will be 120 units. Up to

360 buses can be manufactured each quarter based on each employee working on regular time. Each bus costs $40,000 if manufactured on regular time. It is also possible to manufacture up to 80 buses on overtime. Each bus so produced costs $55,000.

To keep one bus in inventory for one quarter costs $2,000. This charge represents the cost of tied-up capital, warehouse, and insurance expenses. As a buffer against potentially high demand at the end of the planning horizon, the ending inventory should be at least 70 buses.

Buses may be delivered late to the customers, but this comes with a penalty cost of $13,000 per quarter per bus. We wish to formulate a model which seeks to minimize the sum of all costs.

3.7.8 Production Planning 2

A company which produces a single product has definite orders for this product over the next four quarters as follows:

Quarter	1	2	3	4
Demand	350	680	275	590

The company ended the previous year with an inventory of 200 units, and the final quarter production level was 400 units. The company wishes to end this year with an inventory of at least 250 units.

It costs $3.50 per unit to increase the production level from one quarter to the next, and $6.00 per unit to decrease it. The cost of holding inventory from one quarter to the next is $4.80 per unit per quarter. No shortages are permitted. The company wishes to minimize the sum of production level change costs and inventory costs over the four quarter planning horizon.

3.7.9 Production Planning 3

A company has firm orders for the following quantities over the next six months:

Month	1	2	3	4	5	6
Demand	200	300	700	500	100	400

To change the production level from one month to the next costs $2 per unit increased or $5 per unit decreased. To hold a unit in inventory for one month costs $4. No shortages are permitted.

The company starts off (think of this as month 0) with 50 units in inventory. There must be no inventory left over at the end of month six. The previous month's (month 0) production level was 240 units. There is no restriction on the level of production in month six. The company wishes to minimize the sum of production level change costs and inventory holding costs over the six month horizon.

3.7.10 Cutting Stock 1

A lumber yard stocks 1 cm diameter dowels in a standard width of 200 cm. A customer wishes to buy 38 dowels of width 120 cm, 32 dowels of width 45 cm, and sixteen dowels of width 40 cm.

(a) Make a model for this situation.

(b) Find the solution using the Excel Solver.

3.7.11 Cutting Stock 2

A paper mill produces rolls of paper in a standard width of 150 cm. All paper produced has a thickness of 0.08mm, and each roll has a length of 1000 metres. The customers all desire rolls of this thickness and this length. The mill currently has the following non-standard width orders:

Width (cm)	Number
87	51
61	65
45	18

(a) Formulate a model for this problem.

(b) Solve the model using the Excel Solver.

3.8 More Difficult Problems

As these problems might be used for hand-in assignments, solutions are not provided.

3.8.1 Cutting Stock

A paper mill produces rolls of paper in standard widths of 90 cm and 200 cm. All paper produced has a thickness of 0.1 mm, and each roll has a length of 1000 metres. The customers all desire rolls of this thickness and this length, but not necessarily either of the two standard widths. The mill currently has the following non-standard width orders:

Width (cm)	Number
95	25
80	31
46	23
21	68

The paper which is leftover on the cut rolls is re-cycled. Formulate a model which will minimize the amount of paper which needs to be re-cycled.

3.8.2 Production Planning

The production manager of a company needs to determine next month's production plan for the company's ten products. The products use six resources: assembly line 1; assembly line 2; painting; dryers; packaging; and storage. Storage is measured in m^3, and the others are in hours. The requirements for each product are:

	Product										Resource
	1	2	3	4	5	6	7	8	9	10	Available
Assembly 1	2	1	0.5	0.75	1.5	0.25	0	0	0	0	2100
Assembly 2	0	0	0.3	0.45	0.5	0.65	1	0.8	2	3	1500
Painting	0	0.2	0	0.4	0.5	0.65	1.5	0.1	0.15	2	1000
Dryers	0	0.3	0	0.8	0.2	0	1	0.3	0.2	1	1000
Packaging	0.5	0.1	1	0.2	0.1	0.65	0.1	0.2	1	0.5	1600
Storage	0.25	0.1	0.5	0.45	0.4	0.25	0.1	0.1	2	0.3	1300

In addition, there are some company constraints which must be satisfied.

(i) There should be at most 4,500 units produced.

(ii) There should be at least two units of product 3 for every unit of the combined production of products 6 and 8 produced.

(iii) The total production of product 4 should be no more than the combined production of products 2 and 7.

(iv) The combined production of products 1 and 5 must be at most twice the production of product 9.

The profit contribution in dollars per unit for each of the ten products is 2.1, 3.2, 1.6, 4.8, 1.2, 4.3, 3.5, 1.8, 5.5, 3.9 respectively.

For both parts (a) and (b), state the solution so that the production manager will understand it.

(a) Given that the company wants to maximize its profit, define the variables and set up the appropriate algebraic model in standard linear optimization format.

(b) Convert the algebraic model from (a) to a spreadsheet model, and solve it using a spreadsheet solver.

(c) Now suppose that the cost of storage (\$2.50 per m^3) has not been taken into account in the profits given for each product, but we now wish to include it. Modify the models from (a) and (b) (just show what's different compared with (a)), with the amount of storage used becoming the eleventh variable, and solve the modified model using a spreadsheet.

Chapter 4

Sensitivity Analysis

4.1 Introduction

After a model has been solved, it is often desirable to know what would happen if one or more of the parameters of the model were to change. When we do this we say that we are performing a *sensitivity analysis* on the model. One can always answer such a question by re-solving the model with the altered parameters. Sometimes, however, such questions can be answered simply by using some of the information which was determined from solving the initial model. We wish to be able to identify such situations so that unnecessary re-solving on the Excel Solver is avoided. Ideally, the user would have an Excel Solver printout of the solution to the initial model, and would then use this information to answer a set of questions. Only if a question could *not* be answered by using the sensitivity analysis methods of this chapter, would we then run the model again with alterations.

There are three types of sensitivity analysis that we will perform:

1. Changes to the objective function coefficients.

2. Changes to the right-hand side values of the non-binding constraints.

3. Changes to the right-hand side values of the binding constraints.

In this chapter we will consider sensitivity analysis in three contexts. First, for models which have only two decision variables, we will perform a sensitivity analysis graphically. Secondly, we will see sensitivity analysis using an Excel Solver Sensitivity Report for one-at-a-time changes. Finally, we will describe the situations in which the effect of varying two or more coefficients simultaneously can be determined analytically based on the Sensitivity Report. In all situations, sensitivity analysis is not compatible with declaring any of the variables to be integer; for all variables, we must be willing to accept fractional values.

4.2 Graphical Approach to Sensitivity Analysis

4.2.1 Problem Description

Wood Products Limited buys fine hardwoods from around the world from which they make specialized products for the quality furniture market. Two of their products are two types of spindles.

A type 1 spindle requires 6 cuts, then 4 minutes of polishing, followed by 6.5 minutes of varnishing. A type 2 spindle requires 15 cuts, then 4 minutes of polishing, followed by 4.75 minutes of painting. There is one cutting machine which can operate up to 135 cuts per hour. There is one polishing machine – allowing for maintenance it can operate up to 54 minutes per hour. Both the varnish and paint shops can only handle one spindle at a time. Because of a periodic need for high volume ventilation, the varnish and paint shops cannot be operated continuously. These shops are available for production 58.5 and 57 minutes per hour, respectively.

For each type 1 spindle produced, the company obtains a contribution to profit of \$3. For each type 2 spindle produced, the contribution to profit is \$4. How many spindles of each type should be produced each hour so that the total contribution to profit is maximized?

4.2.2 Model

We define:

X_1 — the number of type 1 spindles produced per hour

X_2 — the number of type 2 spindles produced per hour.

As always, each constraint is identified by a word description on the left-hand side. In addition to this, the constraints have been numbered on the right-hand side, to make it easier to reference these constraints in the text which follows later.

$$
\begin{array}{rrcrcrl}
\text{maximize} & 3X_1 & + & 4X_2 & & & \\
\text{subject to} & & & & & & \\
\text{Cutting} & 6X_1 & + & 15X_2 & \leq & 135 & (1) \\
\text{Polishing} & 4X_1 & + & 4X_2 & \leq & 54 & (2) \\
\text{Varnishing} & 6.5X_1 & & & \leq & 58.5 & (3) \\
\text{Painting} & & & 4.75X_2 & \leq & 57 & (4) \\
 & & & & & & \\
 & X_1 & , & X_2 & \geq & 0 &
\end{array}
$$

4.2.3 Graphical Solution

Because of the two 4's in the polishing constraint, this constraint will be on a diagonal. Since it's $\leq$, the arrow will point south-west. So, since $54/4 = 13.5$, having a 14 by 14 grid must contain the optimal solution. Using these boundaries, we obtain:

$$
\begin{array}{rrcrcrlcc}
 & & & & & & & \text{First Point} & \text{Second Point} \\
\text{Cutting} & 6X_1 & + & 15X_2 & \leq & 135 & (1) & (0,9) & (14,3.4) \\
\text{Polishing} & 4X_1 & + & 4X_2 & \leq & 54 & (2) & (0,13.5) & (13.5,0) \\
\text{Varnishing} & 6.5X_1 & & & \leq & 58.5 & (3) & X_1 = 9 & \text{vertical} \\
\text{Painting} & & & 4.75X_2 & \leq & 57 & (4) & X_2 = 12 & \text{horizontal}
\end{array}
$$

The graph (displaying both numerical labels and the names of the constraints) is shown in Figure 4.1.

We see that constraints (1) and (2), i.e. the cutting and polishing constraints, are binding.

Using Algebra

The equations we need to solve are:

$$
\begin{array}{rcl}
6X_1 + 15X_2 & = & 135 \\
4X_1 + 4X_2 & = & 54
\end{array}
$$

Multiplying the second equation by 6/4 = 1.5 we obtain:

$$
\begin{array}{rcl}
6X_1 + 15X_2 & = & 135 \\
6X_1 + 6X_2 & = & 81
\end{array}
$$

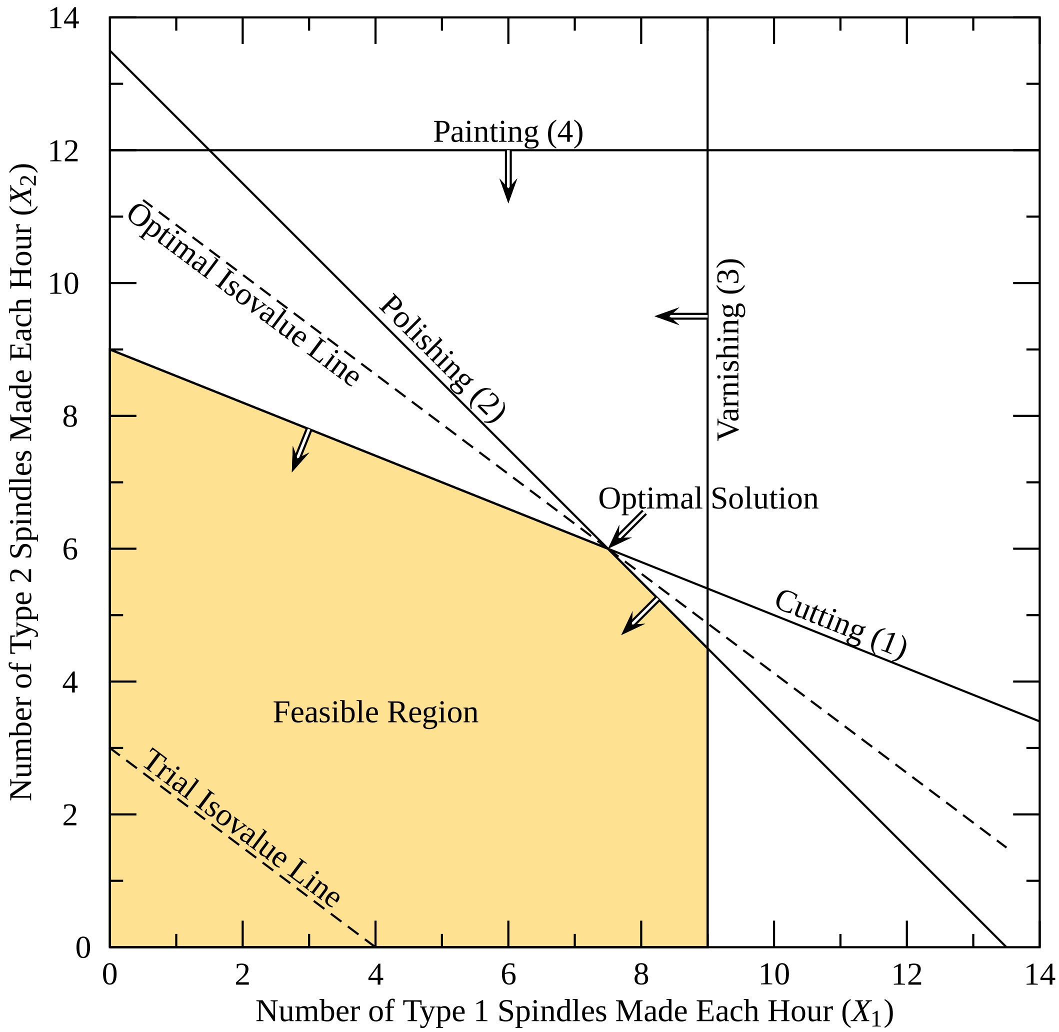


Figure 4.1: Spindle Problem – Original Version

Subtracting the bottom from the top gives $9X_2 = 54$, and hence $X_2 = 6$. Therefore $4X_1 + 4(6) = 54$, hence $4X_1 = 30$, and therefore $X_1 = 7.5$. The 7.5 Type 1 spindles per hour simply means that we must produce 15 of them every two hours, hence the fractional solution is not of concern.

Using Matrix Operations in Excel (Optional)

Alternatively, we could solve the equations using Excel. Beginning with

$$\begin{array}{rrcrcr} \text{Cutting} & 6X_1 & + & 15X_2 & = & 135 \\ \text{Polishing} & 4X_1 & + & 4X_2 & = & 54 \end{array}$$

we convert these equations to matrix form:

$$\begin{bmatrix} 6 & 15 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 135 \\ 54 \end{bmatrix}$$

Using the Excel MINVERSE function to perform the matrix inversion we obtain:

$$\begin{bmatrix} 6 & 15 \\ 4 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} -0.111111 & 0.111111 \\ 0.416667 & -0.166667 \end{bmatrix}$$

Using MMULT to multiply the inverse by the right-hand side values, we obtain:

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 7.5 \\ 6.0 \end{bmatrix}$$

The unique solution is $X_1 = 7.5$, and $X_2 = 6$.

The OFV

The objective function value at the point of optimality is

$$\begin{aligned} \text{OFV}^* &= 3X_1^* + 4X_2^* \\ &= 3(7.5) + 4(6) \\ &= 22.5 + 24 \\ &= 46.5 \end{aligned}$$

4.2.4 Changes to the Objective Function Coefficients

We now repeat the original solution, but consider also a new objective function in which we keep the coefficient of X_1 at its current value of 3, but increase the coefficient of X_2 from 4 to 5. The new objective function is:

$$\text{maximize } \; 3X_1 + 5X_2$$

All the constraints are as they were before, but the dashed lines which show the trial and optimal isovalue lines are modified to obtain the graph shown in Figure 4.2.

The slope of this line is negative, and indeed any two-variable constraint for which both coefficients are positive (which is what happens most of the time) will have a negative slope. Since its easier to deal with positive numbers, for constraints of negative slope we define the *steepness* as the rise over the negative of the run. In terms of the objective function coefficients, where $c_1 \geq 0$ and $c_2 > 0$, the steepness is conveniently found as:

$$\text{steepness } = \frac{c_1}{c_2}$$

We see that this small change to the objective function makes the isovalue line less steep. The steepness of the isovalue line was originally $3/4 = 0.75$; it is now $3/5 = 0.6$. In this example, this reduction in steepness does not change the optimal solution as far as the two decision variables are concerned, because the solution remains at the same corner of the feasible region, where $X_1 = 7.5$ and $X_2 = 6$. Of course, the OFV will increase by \$6.00, because we obtain an extra \$1.00 per unit for the 6 units we make of X_2, i.e. the OFV increases from \$46.50 to \$52.50. Or, we can compute this new value as $3X_1 + 5X_2 = 3 \times 7.5 + 5 \times 6 = 52.5$.

As the coefficient of X_2 is increased, the optimal solution stays at the same corner until the critical value of 7.5 is reached, at which point multiple optima exist. Were we to attempt an even larger increase in the coefficient of X_2, for example a change to "maximize $3X_1 + 8X_2$," then the optimal corner would change, in this case to the point $X_1 = 0$, and $X_2 = 9$.

If we were to *decrease* the coefficient of X_2, then the isovalue line would be steeper than the original one. As this coefficient is decreased the solution stays at the same corner until the critical value of 3 is reached, at which point multiple optima exist. With a further decrease, say to "maximize $3X_1 + 2X_2$," the optimal corner would change to $X_1 = 9$, $X_2 = 4.5$. For each cost coefficient, we wish to determine the critical values between which the optimal solution does not change.

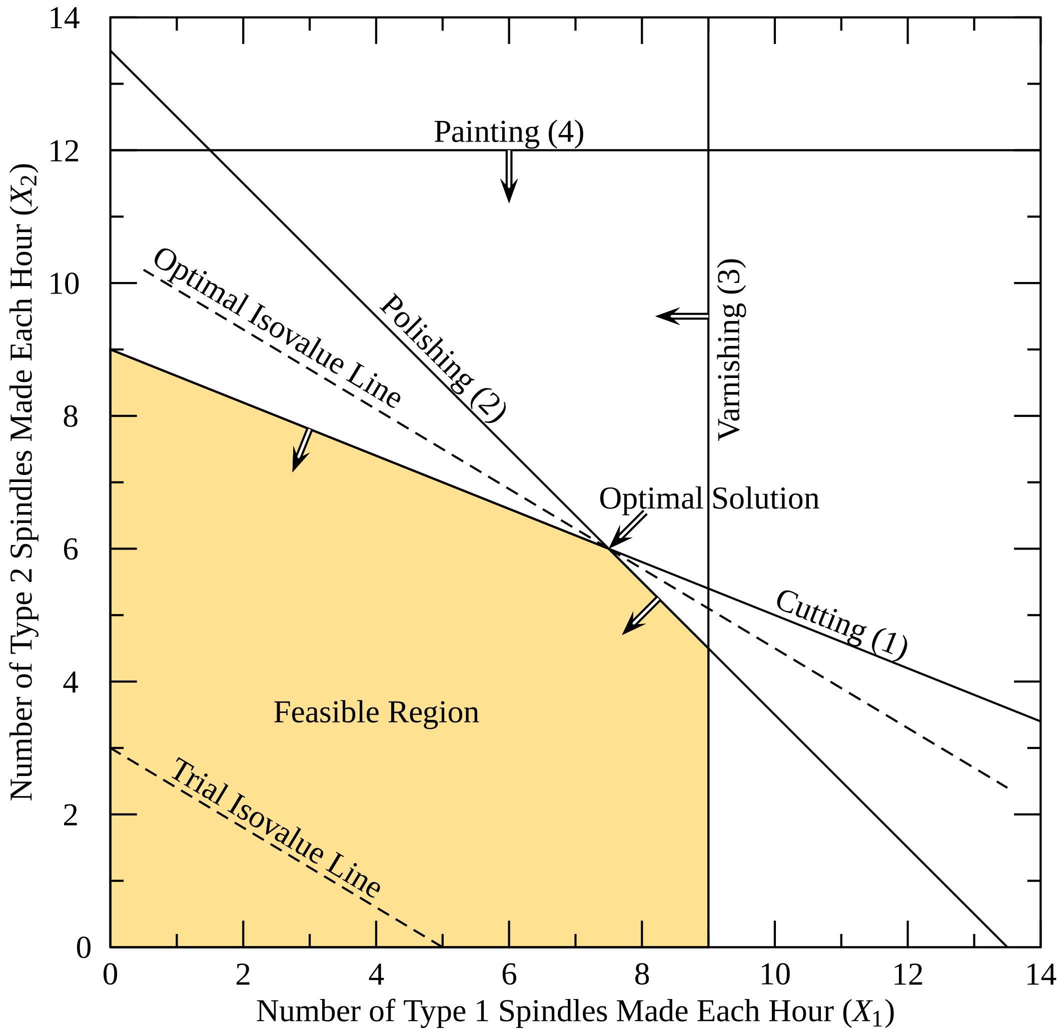


Figure 4.2: Spindle Problem – New Objective: maximize $3X_1 + 5X_2$.

The important thing in determining whether the current corner remains optimal or whether a new optimal corner is obtained, is the relationship between the steepness of the objective function and the steepness of each *binding* constraint. We consider a general objective function for this problem:

$$\max\ c_1X_1 + c_2X_2$$

We are seeking values for c_1 and c_2 which make the isovalue lines have a steepness which is in-between the steepnesses of the two binding constraints, i.e. steeper than the boundary of constraint (1), but not as steep as the boundary of constraint (2). In the discussion which follows, we assume that both coefficients are strictly positive (which makes sense because the company is selling these products in the marketplace).

Since constraint (1) is $6X_1 + 15X_2 \leq 135$, the objective function will be steeper than the boundary of (1) provided that [1]

$$\frac{c_1}{c_2} \geq \frac{6}{15} = 0.4$$

Since constraint (2) is $4X_1 + 4X_2 \leq 54$, the isovalue line will not be as steep as the boundary of (2) provided that

$$\frac{c_1}{c_2} \leq \frac{4}{4} = 1$$

Overall, therefore, the solution remains at the corner where the current two binding constraints (i.e. (1) and (2)) meet provided that $c_1, c_2 > 0$ and

$$0.4 \leq \frac{c_1}{c_2} \leq 1$$

At the current value of $c_2 = 4$, we obtain the range

$$0.4 \leq \frac{c_1}{4} \leq 1$$

[1] Or, we could write:

$$\frac{c_2}{c_1} \leq \frac{15}{6} = 2.5$$

In this example, either of these forms is acceptable, but in general the one in the text above should be used if, at the corner under consideration, the isovalue line could be horizontal, and the form in this footnote should be used if the isovalue line could be vertical. Doing this will prevent a division by 0 problem.

which simplifies to

$$1.6 \leq c_1 \leq 4$$

At the current value of $c_1 = 3$, we obtain the range

$$0.4 \leq \frac{3}{c_2} \leq 1$$

Because $c_2 > 0$ by assumption, when we cross-multiply by c_2 the inequalities remain unchanged. We therefore obtain $0.4c_2 \leq 3$, and $3 \leq c_2$, which simplifies to

$$3 \leq c_2 \leq 7.5$$

In the next section we will see that the Excel Solver computes the ranges for each coefficient assuming that the other coefficient is held constant, just as we just did here. Most software for linear optimization, including the Excel Solver, will speak of the "allowable increase" (AI) or "allowable decrease" (AD) from the current values of the coefficients. Hence with $c_1 = 3$ and $1.6 \leq c_1 \leq 4$, the AI is $4 - 3 = 1$ and the AD is $3 - 1.6 = 1.4$. With $c_2 = 4$ and $3 \leq c_2 \leq 7.5$, we obtain AI = 3.5 and AD = 1.

The general form $0.4 \leq \frac{c_1}{c_2} \leq 1$ is far more useful than what we obtain from the Excel Solver because it allows us to consider two simultaneous changes to the objective function coefficients. We now graph the region for c_1 and c_2 where the particular corner defined by the interception of (1) and (2) remains optimal. This graph is shown in Figure 4.3.

This region is bounded by $c_1 - c_2 \leq 0$ and $c_1 - 0.4c_2 \geq 0$ (which follow from $0.4 \leq \frac{c_1}{c_2} \leq 1$). Within this region, a horizontal line labelled A–B gives the range for c_1 if c_2 is held constant, and a vertical line labelled C–D gives the range for c_2 if c_1 is held constant. The A–B line gives the range 1.6 to 4, and the C–D line gives the range 3 to 7.5, as we saw above.

4.2.5 Changes to the Right-Hand Side Values of the Non-Binding Constraints

The easiest type of sensitivity analysis is that of a change to the right-hand side (rhs) of a *non-binding* constraint. Whenever the rhs of any constraint (binding or non-binding) is changed, the new boundary is parallel with the old one. An increase to the rhs moves the boundary farther from the origin, while a decrease moves it closer.

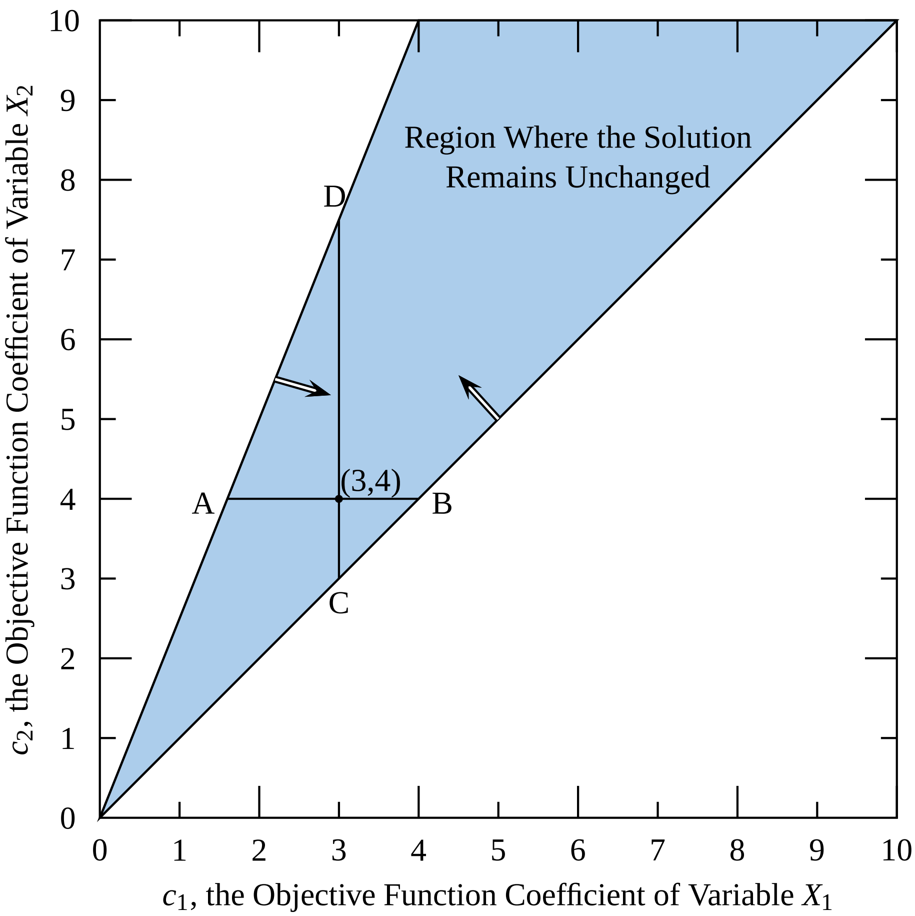


Figure 4.3: Region of (c_1, c_2) Where the Solution Remains Unchanged

In this example, both constraints (3) and (4) are non-binding. Suppose now that we have the original objective function but that the right hand side of constraint (3) is changed. Whenever the rhs is *increased*, the new boundary moves farther away from the optimal solution, and so there is no effect on the optimal corner. We will have the same values for the variables and the objective function value. Mathematically, the right-hand side value can be increased indefinitely, but of course it makes no sense for this number to be more than 60, because it represents the number of operating minutes per hour.

If the rhs is *decreased*, then there is no effect provided that the new boundary does not "chop off" the current corner. For this to happen, the decrease must not exceed the slack on the constraint. The left-hand side value of constraint (3) is $6.5(7.5)+0(6) = 48.75$, and the right-hand side is 58.5, hence the slack is $58.5-48.75 = 9.75$. Hence if the decrease is less than or equal to 9.75 units, we would have the same values for X_1^*, X_2^*, and OFV*. If the right hand side value were to decrease by exactly 9.75 units, then the current corner remains optimal but it would now have three lines passing through it. Such a corner is said to be *degenerate*.

If the decrease is more than 9.75 units, i.e. if the new right hand side of (3) is less than $58.5-9.75 = 48.75$, then (3) would become a binding constraint and (2) would become non-binding. Hence the current solution remains optimal provided that the rhs of (3) is not decreased by more than 9.75 units.

At the optimal solution, the value of the left-hand side of the painting constraint (4) is $0(7.5)+4.75(6) = 28.5$. The slack is therefore $57-28.5 = 28.5$. Therefore, constraint (4) can be decreased by up to 28.5 units without affecting the optimal solution. In general, the rhs of a *non-binding* $\leq$ constraint may be decreased by up to the amount of the slack without affecting the optimal solution; the rhs of such a constraint may be increased indefinitely.[2]

4.2.6 Changes to the Right-Hand Side Values of the Binding Constraints

If the rhs of a *binding* constraint is changed, then the optimal solution will change.[3] However, we shall see that within an "allowable range" (to be determined), the

[2] This is a statement of what is mathematically allowable; it has nothing to do with whether or not it is technologically possible to alter the constraint.

[3] There's one minor exception. If the current optimal solution is degenerate, then the rhs of the "middle" of the three constraints can be increased (for $\leq$) or be decreased (for $\geq$) without changing the solution.

values of the variables and the OFV will vary linearly with changes to the rhs.

Determining the Allowable Range for the Polishing Constraint

To illustrate how we determine the allowable range, we consider an altered polishing constraint. We will denote this altered constraint as $(2')$:

$$\text{Polishing} \qquad 4X_1 + 4X_2 \leq 48 \quad (2')$$

We now re-solve the model. The first change is that the new polishing constraint is parallel with the old one, but closer to the origin. This causes the feasible region to be smaller than it was before. The part of the former feasible region which is now infeasible is shown in blue. The optimal solution moves from its former location along the cutting constraint (as shown by the arrows) until it reaches the new interception point of the cutting and polishing constraints ((1) and $(2')$). The new solution is shown in Figure 4.4.

We have the same corner as before, but the corner itself has moved. We determine this new solution by solving the set of equations:

$$\begin{aligned} 6X_1 + 15X_2 &= 135 \\ 4X_1 + 4X_2 &= 48 \end{aligned}$$

We solve these equations either by algebra or by using Excel to obtain $X_1^* = 5$, and $X_2^* = 7$. If done on Excel, and if we already found the inverse matrix using MINVERSE as shown on page 157, all we need do is use MMULT to multiply the already-found inverse by the new right-hand side. We then compute $\text{OFV}^* = 3 \times 5 + 4 \times 7 = 43$.

When we say that "we have the same corner as before," we are saying that we have the same pair of binding constraints. Whether the rhs of the second constraint is the original value of 54 or the new value of 48, the binding constraints are cutting and polishing. If we were to keep decreasing the rhs, we would eventually reach the point where the cutting constraint meets the vertical axis. At this point we would have a degenerate optimal solution, since three lines in two-dimensional space consisting of constraints (1), $(2')$, and the non-negativity restriction $X_1 \geq 0$ (i.e. the vertical axis) all pass through the optimal solution. Either by inspection or by solving $6X_1 + 15X_2 = 135$ at $X_1 = 0$, we obtain $X_2 = 9$. Putting these values into the objective function we obtain $\text{OFV} = 3(0) + 4(9) = 36$. Hence this solution is $X_1^* = 0$, $X_2^* = 9$, $\text{OFV}^* = 36$.

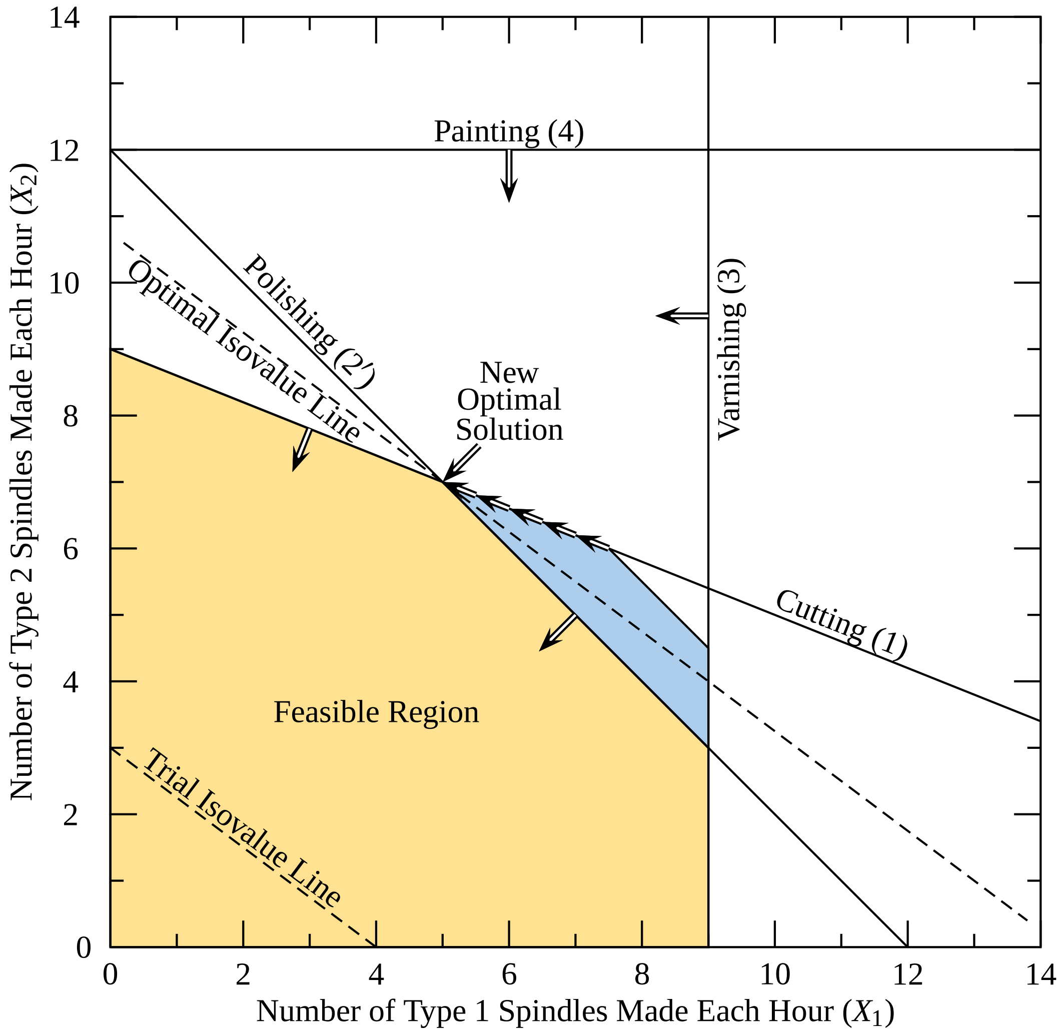


Figure 4.4: Polishing Constraint with Altered RHS: $4X_1 + 4X_2 \leq 48 \quad (2')$

We now substitute these values into the left-hand side of the polishing constraint, to obtain $4(0)+4(9)=36$. [4] Hence 36 is the lowest value for the rhs of the polishing constraint which makes the cutting and polishing constraints remain binding.

A further decrease in the rhs of constraint (2) would cause constraint (1) to become non-binding. The binding constraints would then consist of the modified (2) and the vertical axis, and would remain as such for any positive value of the rhs of constraint (2). If the rhs were to fall to 0, then only the origin would be feasible (and hence optimal). If the rhs were to fall below zero, then this model would no longer have a feasible solution.

If we were to *increase* the rhs of constraint (2) from its current value of 54, then the constraints (1) and (2) would remain binding until the modified constraint (2) intercepts the interception point of constraints (1) and (3), thereby creating a point of degeneracy. Constraints (1) and (3) intercept where:

$$\begin{aligned} 6X_1+15X_2 &= 135 \\ 6.5X_1 &= 58.5 \end{aligned}$$

From the second equation we obtain $X_1=9$, and hence the first equation becomes

$$6(9)+15X_2=135$$

and solving we obtain $X_2=5.4$. Hence the critical value for the rhs of constraint (2) which results in degeneracy at this corner is

$$4X_1+4X_2=4\times 9+4\times 5.4=57.6$$

For an increase in the rhs of (2) beyond 57.6, constraint (2) would become redundant, and the optimal solution would remain at $X_1=9$, $X_2=5.4$.

Hence, the range for the rhs of constraint (2) (denoted as b_2) for which the current binding constraints continue as such is

$$36\leq b_2\leq 57.6$$

Alternatively, one can say that compared with the current value of $b_2=54$, the rhs has an "allowable increase" of 3.6 and an "allowable decrease" of 18. In this context, the word *allowable* means the maximum change for which the binding constraints remain unchanged; it has nothing to do with permission to alter a constraint.

[4] It is of course just a coincidence that the numerical value for the rhs of constraint (2) and the OFV are the same. In any case, the units are different.

The Values of the Variables

Let Δb_2 be the change (positive or negative) to the rhs value of constraint (2), i.e. the new rhs value is $54 + \Delta b_2$. The equations from the binding constraints are now:

$$\begin{aligned} 6X_1 + 15X_2 &= 135 \\ 4X_1 + 4X_2 &= 54 + \Delta b_2 \end{aligned}$$

Multiplying the second constraint by 6/4 = 1.5 we obtain:

$$\begin{aligned} 6X_1 + 15X_2 &= 135 \\ 6X_1 + 6X_2 &= 81 + 1.5\Delta b_2 \end{aligned}$$

Subtracting the bottom from the top gives:

$$9X_2 = 54 - 1.5\Delta b_2$$

and therefore

$$X_2 = 6 - \frac{\Delta b_2}{6}$$

Substituting this expression into the first equation gives:

$$\begin{aligned} 6X_1 + 15X_2 &= 135 \\ 6X_1 + 15\left(6 - \tfrac{1}{6}\Delta b_2\right) &= 135 \\ 6X_1 + 90 - 2.5\Delta b_2 &= 135 \\ 6X_1 &= 45 + 2.5\Delta b_2 \\ X_1 &= 7.5 + \tfrac{5}{12}\Delta b_2 \end{aligned}$$

Alternatively, we could have substituted the expression into the second equation, but this would have to include the Δb_2 on the right-hand side.

$$\begin{aligned} 4X_1 + 4X_2 &= 54 + \Delta b_2 \\ 4X_1 + 4\left(6 - \tfrac{1}{6}\Delta b_2\right) &= 54 + \Delta b_2 \\ 4X_1 + 24 - \tfrac{4}{6}\Delta b_2 &= 54 + \Delta b_2 \\ 4X_1 &= 30 + \tfrac{5}{3}\Delta b_2 \\ X_1 &= 7.5 + \tfrac{5}{12}\Delta b_2 \end{aligned}$$

Note that the Excel approach of using MINVERSE and MMULT does not help us here. This is because Excel gives a numerical solution, but what we need here is an analytical expression.

The OFV and the Shadow Price

We substitute the expressions for X_1 and X_2 into the equation for the OFV:

$$\begin{aligned}
\text{OFV} &= 3X_1 + 4X_2 \\
&= 3(7.5 + \tfrac{5}{12}\Delta b_2) + 4(6 - \tfrac{1}{6}\Delta b_2) \\
&= 22.5 + \tfrac{15}{12}\Delta b_2 + 24 - \tfrac{4}{6}\Delta b_2 \\
&= 46.5 + \tfrac{7}{12}\Delta b_2
\end{aligned}$$

For each increase/decrease of one unit in the rhs of (2), the objective will increase/decrease by $\frac{7}{12} = 0.583333...$ or about 58.3 cents, provided that we remain within the allowable range. This rate of change of the OFV per unit change in the rhs is called the *shadow price*. [5]

The shadow price can also be found numerically, by computing the OFV at the lower or upper limit of the allowable range and then comparing this with its original value. At the allowable decrease of 18, the altered (2) intercepts (1) and the vertical axis, and the solution is $X_1 = 0$, $X_2 = 9$, and $\text{OFV} = 3(0) + 4(9) = 36$. Hence as the rhs falls by 18, the OFV falls by $46.5 - 36 = 10.5$. The shadow price is therefore $-10.5/-18 = 0.583333....$ Or, we can use the upper limit. At the allowable increase of 3.6, the altered (2) intercepts (1) and (3), and the solution is $X_1 = 9$, $X_2 = 5.4$, and $\text{OFV} = 3(9) + 4(5.4) = 48.6$. Hence the shadow price is $(48.6 - 46.5)/3.6 = 0.583333....$

Determining the Allowable Range for the Cutting Constraint

We now perform a similar analysis for changes to the right hand side of the cutting constraint (constraint (1)), denoted as b_1. For a decrease, we see that constraints (1) and (2) remain binding until (and including) the point where the boundaries of constraints (1) (modified), (2), and (3) intercept. Constraints (2) and (3) intercept at $X_1 = 9$, $X_2 = 4.5$. Hence the critical lower value for b_1 is

$$6X_1 + 15X_2 = 6 \times 9 + 15 \times 4.5 = 121.5$$

For an increase to b_1, we see that constraints (1) and (2) remain binding until and including the point where the boundaries of constraints (1) (modified), (2),

[5] The term *dual price*, which is used by LINGO (Appendix A) and other software for linear optimization, is a related but different concept. The dual price gives the *improvement* to the OFV per unit change to the rhs. Hence the shadow price and the dual price are the same for maximization problems, and are equal in magnitude but opposite in sign for minimization problems.

and (4) intercept. Constraints (2) and (4) intercept at $X_1 = 1.5$, $X_2 = 12$. Hence the critical upper value for b_1 is

$$6X_1 + 15X_2 = 6 \times 1.5 + 15 \times 12 = 189$$

Overall, therefore, the binding constraints remain unchanged provided that

$$121.5 \leq b_1 \leq 189$$

Equivalently, compared with the current value of $b_1 = 135$, there is an allowable decrease of 13.5, and an allowable increase of 54.

The Values of the Variables

Let Δb_1 be the change (positive or negative) to the rhs value of constraint (1), i.e. the new rhs value is $135 + \Delta b_1$. The equations from the binding constraints are now:

$$\begin{aligned} 6X_1 + 15X_2 &= 135 + \Delta b_1 \\ 4X_1 + 4X_2 &= 54 \end{aligned}$$

Multiplying the second constraint by 6/4 = 1.5 we obtain:

$$\begin{aligned} 6X_1 + 15X_2 &= 135 + \Delta b_1 \\ 6X_1 + 6X_2 &= 81 \end{aligned}$$

Subtracting the bottom from the top gives:

$$9X_2 = 54 + \Delta b_1$$

and therefore

$$X_2 = 6 + \frac{\Delta b_1}{9}$$

Substituting this expression into the first equation gives:

$$\begin{aligned} 6X_1 + 15X_2 &= 135 + \Delta b_1 \\ 6X_1 + 15\left(6 + \tfrac{1}{9}\Delta b_1\right) &= 135 + \Delta b_1 \\ 6X_1 + 90 + \tfrac{15}{9}\Delta b_1 &= 135 + \Delta b_1 \\ 6X_1 &= 45 - \tfrac{6}{9}\Delta b_1 \\ X_1 &= 7.5 - \tfrac{1}{9}\Delta b_1 \end{aligned}$$

Alternatively, we could use the second equation, which does not have a Δb_1 term on the right-hand side:

$$\begin{aligned}
4X_1 + 4X_2 &= 54 \\
4X_1 + 4\left(6 + \tfrac{1}{9}\Delta b_1\right) &= 54 \\
4X_1 + 24 + \tfrac{4}{9}\Delta b_1 &= 54 \\
4X_1 &= 30 - \tfrac{4}{9}\Delta b_1 \\
X_1 &= 7.5 - \tfrac{1}{9}\Delta b_1
\end{aligned}$$

The OFV and the Shadow Price

We substitute the expressions for X_1 and X_2 into the equation for the OFV:

$$\begin{aligned}
\text{OFV} &= 3X_1 + 4X_2 \\
&= 3(7.5 - \tfrac{1}{9}\Delta b_1) + 4(6 + \tfrac{1}{9}\Delta b_1) \\
&= 22.5 - \tfrac{3}{9}\Delta b_1 + 24 + \tfrac{4}{9}\Delta b_1 \\
&= 46.5 + \tfrac{1}{9}\Delta b_1
\end{aligned}$$

The shadow price within the allowable range is therefore $\frac{1}{9}$ = 0.11111... or about 11.1 cents. As before, the shadow price can also be found numerically, by computing the OFV at the lower or upper limit of the allowable range and then comparing this with its original value.

4.3 The Solver Sensitivity Report

Sensitivity analysis by using graphical analysis is tedious and of course, limited to two variables. In practice, a model is solved using a computer, and when we do this when using the Excel Solver we ask for the Sensitivity Report to be created. The equivalent procedure for LINGO is described in section A.1.3. We first illustrate this using the Wood Products example that we have just completed.

4.3.1 Wood Products Example

We recall that the algebraic model for Wood Products is:

X_1 — the number of type 1 spindles produced per hour

X_2 — the number of type 2 spindles produced per hour.

$$
\begin{array}{rrcrcrl}
\text{maximize} & 3X_1 & + & 4X_2 & & & \\
\text{subject to} & & & & & & \\
\text{Cutting} & 6X_1 & + & 15X_2 & \leq & 135 & (1) \\
\text{Polishing} & 4X_1 & + & 4X_2 & \leq & 54 & (2) \\
\text{Varnishing} & 6.5X_1 & & & \leq & 58.5 & (3) \\
\text{Painting} & & & 4.75X_2 & \leq & 57 & (4) \\
& & & & & & \\
& X_1 & , & X_2 & \geq & 0 &
\end{array}
$$

When this is put into Excel it appears in formula view as:

	A	B	C	D	E	F
1		**Wood**	**Products**			
2	OFV	X1	X2			
3	=SUMPRODUCT(B4:C4,B5:C5)	Type 1	Type 2			
4	Maximize	3	4			
5	Spindles/Hour					
6						
7	Constraints					RHS
8	Cutting	6	15	=SUMPRODUCT($B$5:$C$5,B8:C8)	<=	135
9	Polishing	4	4	=SUMPRODUCT($B$5:$C$5,B9:C9)	<=	54
10	Varnishing	6.5	0	=SUMPRODUCT($B$5:$C$5,B10:C10)	<=	58.5
11	Painting	0	4.75	=SUMPRODUCT($B$5:$C$5,B11:C11)	<=	57

In normal view the calculated cells are all zeroes because the variable cells are blank:

	A	B	C	D	E	F
1		**Wood**	**Products**			
2	OFV	X1	X2			
3	0	Type 1	Type 2			
4	Maximize	3	4			
5	Spindles/Hour					
6						
7	Constraints					RHS
8	Cutting	6	15	0	<=	135
9	Polishing	4	4	0	<=	54
10	Varnishing	6.5	0	0	<=	58.5
11	Painting	0	4.75	0	<=	57

After we fill in all the required information on the Solver Parameters box and then touch the "Solve" button, we are told that an optimal solution has been found. Before clicking to say that we want to keep it, we click on the Answer report and the Sensitivity Report buttons.

The Excel file in normal view is now:

	A	B	C	D	E	F
1		**Wood**	**Products**			
2	OFV	X1	X2			
3	46.5	Type 1	Type 2			
4	Maximize	3	4			
5	Spindles/Hour	7.5	6			
6						
7	Constraints					RHS
8	Cutting	6	15	135	<=	135
9	Polishing	4	4	54	<=	54
10	Varnishing	6.5	0	48.75	<=	58.5
11	Painting	0	4.75	28.5	<=	57

The Answer Report is:

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$A$3	OFV	0	46.5

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$5	Spindles/Hour Type 1	0	7.5	Contin
$C$5	Spindles/Hour Type 2	0	6	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$D$8	Cutting	135	$D$8<=$F$8	Binding	0
$D$9	Polishing	54	$D$9<=$F$9	Binding	0
$D$10	Varnishing	48.75	$D$10<=$F$10	Not Binding	9.75
$D$11	Painting	28.5	$D$11<=$F$11	Not Binding	28.5

The Answer report gives us all the information that we found when we solved this model graphically. Now we look at the Sensitivity Report:

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Spindles/Hour Type 1	7.5	0	3	1	1.4
$C$5	Spindles/Hour Type 2	6	0	4	3.5	1

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$8	Cutting	135	0.111111111	135	54	13.5
$D$9	Polishing	54	0.583333333	54	3.6	18
$D$10	Varnishing	48.75	0	58.5	1E+30	9.75
$D$11	Painting	28.5	0	57	1E+30	28.5

There are two things that are new to us:

1. Excel's use of scientific notation was mentioned on page 5. The expression `1E+30`, which literally means a 1 followed by 30 zeros, is Excel's way of saying "infinite".

2. The "Reduced Cost" is a term originally developed for minimization models. For a variable not currently in the solution (i.e. its value is 0) the reduced cost is the amount by which the coefficient of that variable must be reduced in order to make that variable > 0. It is not applicable in this example because both variables are in the solution.

4.3.2 Using the Sensitivity Report

When considering any proposed change to a coefficient, one needs to first determine whether the proposed change falls inside or outside the allowable range. When a change is within the allowable range, the effect on the objective function value is always easy to compute. When it falls outside this range, we are limited in what we can conclude.

Within the Allowable Range

Objective Function Coefficients When an objective function coefficient is changed within the allowable range, there is no change to the variables, but the OFV will change because the coefficient has changed. For example, suppose that the selling price of type 2 spindles increases by \$2.00 per spindle. This is less than the allowable increase of \$3.50. We are selling six type 2 spindles. Hence the OFV goes up by \$2.00(6) = \$12.00.

Non-Binding Constraints When the right-hand side value of a non-binding constraint is changed within the allowable range, there is no change to the variables, and no change to the OFV. For example, suppose that we are considering lowering the rhs value of the varnishing constraint by 5 minutes. This is less than the allowable decrease of 9.75 minutes. Hence we obtain the same solution, and the OFV remains unchanged.

Binding Constraints When the right-hand side value of a binding constraint is changed within the allowable range, there will be a change to the variables (but we cannot tell how from the sensitivity report), and the OFV will change by the product of the shadow price and the change to the rhs value. For example, suppose that the cutting constraint's rhs is increased from 135 to 162. This increase of 27 units is allowable ($27 \leq 54$). The shadow price on the cutting constraint is 0.11111111. Therefore, the OFV will increase by 0.111111111(27) = \$3.00.

Outside the Allowable Range

Objective Function Coefficients When an objective function coefficient is changed beyond the allowable range, the variables will change (but we cannot predict how from the sensitivity report). The OFV will change by at least what it would have changed had we not gone beyond the allowable range. For example, suppose that the selling price of type 2 spindles increases by \$4.00 per spindle. This is beyond the allowable increase of \$3.50. We are selling six type 2 spindles. Therefore the increase to the OFV will be at least \$3.50(6) = \$21.00.

The Possibility of Infeasibility When the rhs of a constraint is changed beyond the allowable range, it might increase the size of the feasible region, leave the feasible region unchanged, or decrease the size of the feasible region. The first two cases are not a problem. However, a decrease could be problem – it's possible that if the decrease were large enough that it could entirely eliminate the feasible region. In the next two paragraphs we must assume that this is not happening, or else interpret the OFV of an infeasible model to be $-\infty$ for a maximization model or ∞ for a minimization model.

Non-Binding Constraints When the right-hand side value of a non-binding constraint is changed beyond the allowable range, both the variables and the OFV will change, but we cannot predict either of these things numerically. However, we can say that the OFV will be impaired. In other words, the OFV will decrease for a maximization model, or increase for a minimization model. For example, suppose that we are considering lowering the rhs value of the varnishing constraint by 10 minutes. This is more than the allowable decrease of 9.75 minutes. Hence the OFV will decrease (but we cannot predict by how much).

Binding Constraints When the right-hand side value of a binding constraint is changed beyond the allowable range, the OFV will change by at least what it would have changed had we not gone beyond the allowable range. For example, suppose that the cutting constraint's rhs is increased from 135 to 200. This increase of 65 units is beyond the allowable increase of 54. The shadow price is 0.11111111. Therefore, the OFV will increase by at least 0.111111111(54) = $6.00.

4.3.3 Example 1: Maximization

A chemical laboratory can make three types of chemical powders. The variables X_1, X_2, and X_3 represent the number of kilograms per day of the three chemicals. The chemical company has made the following profit-maximization model in Excel.

	A	B	C	D	E	F	G
1		**Chemical Laboratory Model**					
2	OFV	X1	X2	X3			
3	0	Chemical 1	Chemical 2	Chemical 3			
4	Maximize	32	25	18			
5	kg/day						
6							
7	Constraints						RHS
8	Conveyor	3	5	7	0	<=	550
9	Shipping	5	6	3	0	<=	800
10	Production	2	4	8	0	>=	360
11	Mixing	8	9	4	0	<=	880

We use the Solver to obtain:

	A	B	C	D	E	F	G
1		**Chemical Laboratory Model**					
2	OFV	X1	X2	X3			
3	3600	Chemical 1	Chemical 2	Chemical 3			
4	Maximize	32	25	18			
5	kg/day	90	0	40			
6							
7	Constraints						RHS
8	Conveyor	3	5	7	550	<=	550
9	Shipping	5	6	3	570	<=	800
10	Production	2	4	8	500	>=	360
11	Mixing	8	9	4	880	<=	880

The Answer Report is:

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$A$3	OFV	0	3600

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$5	kg/day Chemical 1	0	90	Contin
$C$5	kg/day Chemical 2	0	0	Contin
$D$5	kg/day Chemical 3	0	40	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$E$10	Production	500	$E$10>=$G$10	Not Binding	140
$E$11	Mixing	880	$E$11<=$G$11	Binding	0
$E$8	Conveyor	550	$E$8<=$G$8	Binding	0
$E$9	Shipping	570	$E$9<=$G$9	Not Binding	230

We see that the solution is to produce 90 kg (all units are per day) of chemical 1, none of chemical 2, and 40 kg of chemical 3. The solution variables are therefore X_1 and X_3. The profit obtained using this production plan is \$3600. The conveyor and mixing constraints are binding.

The Sensitivity Report is:

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	kg/day Chemical 1	90	0	32	4	11.86046512
$C$5	kg/day Chemical 2	0	-11.59090909	25	11.59090909	1E+30
$D$5	kg/day Chemical 3	40	0	18	56.66666667	2

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$E$10	Production	500	0	360	140	1E+30
$E$11	Mixing	880	3.863636364	880	389.2307692	565.7142857
$E$8	Conveyor	550	0.363636364	550	990	110
$E$9	Shipping	570	0	800	1E+30	230

Changes to the Objective Function Coefficients

We consider what happens to the OFV in each of the following situations:

1. The price [per kg] of powder 1: (a) decreases by \$10; (b) increases by \$5.
2. The price of powder 2: (a) decreases by \$18; (b) increases by \$9; (c) increases by \$15.
3. The price of powder 3: (a) increases by \$30; (b) decreases by \$7.

Powder 1 The amount of powder 1 made and sold is represented by variable X_1, which is a solution variable. The price per kg is the *coefficient* of this variable, which is currently \$32 (don't confuse this with the value of the variable itself, which is 90 kg). From the "Variable Cells" section of the sensitivity report, we see that the allowable increase is 4, and the allowable decrease is 11.86046512. In other words, we would obtain the same solution even if the coefficient were to rise from 32 to $32+4=36$, or if it were to fall to $32-11.86046512=20.13953488$. Hence a decrease of \$10 (which is ≤ 11.86046512) would have no effect on the solution; they would still make 90 kg per day of powder 1, and 40 kg per day of powder 3. However, the OFV would decrease by \$10(90) = \$900, i.e. it would fall from \$3600 to \$2700. We could also state this as Δ OFV $= -\$900$. A rise of \$5 (> 4) is beyond the allowable increase, so we would obtain a new solution, and we therefore cannot predict the new value of the OFV exactly. We would have to re-run the model on the Solver replacing the 32 with 37, if we wanted to know the new value exactly. However, we can state that the new OFV will be at least what it would be based on the allowable increase. An increase of 4 would cause the profit to increase by \$4(90) = \$360, hence an increase of 5 would cause an increase of at least this much, i.e. Δ OFV $\geq \$360$.

Powder 2 Variable X_2 is not in the solution; the current price of \$25 per kg isn't high enough to justify making any quantity of powder 2. Ordinary logic therefore tells us that a price *decrease* is not going to change anything; a decrease in the price of \$18 per kg does not change either the solution or the OFV. Note that the allowable decrease is infinite. For a price increase, we cannot determine what will happen by logic – we need to look at the allowable increase from the sensitivity report. This figure is seen to be 11.59090909. Hence an increase of \$9 per kg is less than the allowable increase, and there would be no change to the solution. Furthermore, there would be no change to the OFV, because we are not making any powder 2. If however the price were to rise by \$15 per kg, this would surpass the allowable increase. The solution would change, and the OFV would increase, but neither of these things could be quantified without re-running the model.

Powder 3 The amount of powder 3 made and sold is represented by variable X_3. The current coefficient of this solution variable is \$18. We see from the printout that the allowable increase is $56\frac{2}{3}$, and the allowable decrease is 2. In other words, we would obtain the same solution even if the coefficient were to fall from 18 to $18-2=16$, or if it were to rise to $18+56\frac{2}{3}=74\frac{2}{3}$. Hence an increase of \$30 (which is $\leq 56\frac{2}{3}$) would have no effect on the solution; they would still make 90 kg per day of powder 1, and 40 kg per day of powder 3. However, the OFV would increase by \$30(40) = \$1200. A decrease of \$7 (> 2) is beyond the allowable decrease, so we would obtain a new solution. The new OFV will be at most what it would be based on the allowable decrease. A decrease of 2 in the rhs would cause the profit to decrease by \$2(40) = \$80, hence a decrease of 7 would cause a decrease of at least this much. We must be careful with the inequality here; the *magnitude* is at least \$80. Hence if the change is say \$80 or more downwards, then Δ OFV $\leq -\$80$.

Changes to the Right-Hand-Side Values

We consider what happens to the OFV in each of the following situations:

1. The right-hand side value (rhs) of the conveyor constraint: (a) decreases by 100; (b) decreases by 480; (c) increases by 550; (d) increases by 1200.

2. The rhs of the shipping constraint: (a) decreases by 100; (b) increases by 200; (c) decreases by 300.

3. The rhs of the minimum production constraint: (a) increases by 150; (b) increases by 110.

4. The rhs of the mixing constraint: (a) decreases by 330; (b) decreases by 600; (c) increases by 275.

Conveyor Since the conveyor constraint is binding, any change to the rhs will affect the solution. While the new solution is not easily found without re-running the model, the change to the OFV is easy to predict within the allowable range. We see from the sensitivity report that this constraint has an allowable increase of 990 and an allowable decrease of 110. Hence a decrease of 100 is within the allowable range. To see the effect on the OFV, we need the shadow price for this constraint, which is 0.363636364. The OFV will therefore change by:

$$\begin{aligned} \Delta\text{ OFV} &= \text{(shadow price) } (\Delta\text{ rhs}) \\ &= 0.363636364(-100) \\ &\approx -36.36 \end{aligned}$$

(Note: we can say that the *change* is -36.36, or the *decrease* is 36.36.) If we want the new OFV this is $3600 - 36.3636 = 3563.64$. A decrease of 480 would be beyond the allowable decrease of 110. The decrease in the OFV would therefore be at least 0.363636364(110) = \$40, or we could write Δ OFV $\leq -\$40$. An increase of 550 would be allowable, and would cause the OFV to increase by 0.363636364(550) = \$200. An increase of 1200 would exceed the allowable increase of 990, so the OFV would increase by at least 0.363636364(990) = \$360.

Shipping and Minimum Production The shipping and minimum production constraints are non-binding, so the sensitivity analysis is very easy. If the proposed change is within the allowable range, then there is no change to the OFV. If the proposed change is beyond this range, then the OFV will be impaired, i.e. it would decline for a maximization model. The rhs of the shipping constraint can be increased indefinitely or be decreased by up to 230. Hence a decrease of 100 or an increase of 200 would not affect the OFV. A decrease of 300 would cause the OFV to decrease, though we cannot predict by how much. The minimum production constraint has an allowable increase of 140, and it can be decreased indefinitely. An increase of 150 would cause the OFV to fall; an increase of 110 would leave it unchanged.

Mixing Finally, the mixing constraint is binding. It has an allowable increase of 389.2307692, and an allowable decrease of 565.7142857. The shadow price on the mixing constraint is 3.863636364. Hence a decrease of 330 is within the allowable range and the OFV will fall by 3.863636364(330) = 1275.00. A decrease of 600 would be beyond the allowable range; the OFV would fall by at least 3.863636364(565.714294) = 2185.71. An increase of 275 would be within the allowable range, and the OFV would increase by 3.863636364(275) = 1062.50.

4.3.4 Example 2: Minimization

A company buys food products from some or all of five suppliers. These are mixed together. The mixture must meet minimum requirements for three nutrients, have no more than a specific amount of fat, and then be packed into 14.4 kg bags. The variables have been defined as the amount of input of each of the five suppliers that goes into one bag of mixed product, and are denoted as X_1 to X_5. In this example the objective function coefficients are costs rather than revenues.

The company has made the following cost minimization model in Excel:

	A	B	C	D	E	F	G	H	I
1		**Nutritional Requirements Model**							
2	OFV	X1	X2	X3	X4	X5			
3	0	Food 1	Food 2	Food 3	Food 4	Food 5			
4	Minimize	3.7	8.3	5.1	2.9	3.1			
5	kg								
6									
7	Constraints								RHS
8	Nutrient 1	3	4	6	5	2	0	>=	40.5
9	Nutrient 2	8	6	2	3	5	0	>=	81.0
10	Nutrient 3	4	5	8	7	3	0	>=	54.9
11	Fat	5	3	5	6	4	0	<=	64.8
12	Mass Balance	1	1	1	1	1	0	=	14.4

By using the Solver we obtain:

	A	B	C	D	E	F	G	H	I
1		**Nutritional Requirements Model**							
2	OFV	X1	X2	X3	X4	X5			
3	49.94	Food 1	Food 2	Food 3	Food 4	Food 5			
4	Minimize	3.7	8.3	5.1	2.9	3.1			
5	kg	4.7	0	1.3	0.6	7.8			
6									
7	Constraints								RHS
8	Nutrient 1	3	4	6	5	2	40.5	>=	40.5
9	Nutrient 2	8	6	2	3	5	81	>=	81.0
10	Nutrient 3	4	5	8	7	3	56.8	>=	54.9
11	Fat	5	3	5	6	4	64.8	<=	64.8
12	Mass Balance	1	1	1	1	1	14.4	=	14.4

The Answer Report is:

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$A$3	OFV	0	49.94

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$5	kg Food 1	0	4.7	Contin
$C$5	kg Food 2	0	0	Contin
$D$5	kg Food 3	0	1.3	Contin
$E$5	kg Food 4	0	0.6	Contin
$F$5	kg Food 5	0	7.8	Contin

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$G$11	Fat	64.8	$G$11<=$I$11	Binding	0
$G$12	Mass Balance	14.4	$G$12=$I$12	Binding	0
$G$8	Nutrient 1	40.5	$G$8>=$I$8	Binding	0
$G$9	Nutrient 2	81	$G$9>=$I$9	Binding	0
$G$10	Nutrient 3	56.8	$G$10>=$I$10	Not Binding	1.9

The optimal solution is for each bag of product to be composed of 4.7 kg from supplier 1, none from supplier 2, 1.3 kg from supplier 3, 0.6 kg from supplier 4, and 7.8 kg from supplier 5. The cost of the optimal mixture is \$49.94. All constraints except the one for Nutrient 3 are binding.

The Sensitivity Report is:

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	kg Food 1	4.7	0	3.7	1.657142857	1.12
$C$5	kg Food 2	0	1.288888889	8.3	1E+30	1.288888889
$D$5	kg Food 3	1.3	0	5.1	0.828571429	2.327272727
$E$5	kg Food 4	0.6	0	2.9	1.706666667	0.773333333
$F$5	kg Food 5	7.8	0	3.1	1.866666667	8.533333333

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$G$11	Fat	64.8	-1.422222222	64.8	2.127272727	0.72
$G$12	Mass Balance	14.4	5.055555556	14.4	0.327272727	1.017391304
$G$8	Nutrient 1	40.5	1.088888889	40.5	1.8	1.71
$G$9	Nutrient 2	81	0.311111111	81	3.6	16.92
$G$10	Nutrient 3	56.8	0	54.9	1.9	1E+30

Changes to the Objective Function Coefficients

We consider what happens to the OFV in each of the following situations:

1. The cost [per kg] from supplier 1: (a) increases by \$1.50; (b) decreases by \$2.50.

2. The cost from supplier 2: (a) decreases by \$1.28; (b) increases by \$50.

3. The cost from supplier 3: (a) increases by 30 cents; (b) increases by 85 cents; (c) decreases by \$2.

4. The cost from supplier 4: (a) decreases by 50 cents; (b) decreases by \$1.00; increases by \$1.50.

5. The cost from supplier 5: (a) decreases by \$2; (b) increases by \$5.

Supplier 1 Doing a sensitivity analysis on the objective function coefficients is no different for minimization than it is for maximization. The company is currently paying \$3.70 per kg to purchase 4.7 kg (per bag of finished product) from

Supplier 1; there is an allowable increase of 1.657142857 and an allowable decrease of 1.12. An increase of \$1.50 is within the allowable range, and the OFV would increase by \$1.50(4.7) = \$7.05. A decrease of \$2.50 is outside the allowable range, so the OFV would fall by at least \$1.12(4.7) = \$5.264.

Supplier 2 The coefficient of X_2 can be increased indefinitely or be decreased by just under \$1.29 (1.288888889). Hence a decrease of \$1.28 or an increase of \$50 are both within the allowable range, and since X_2 is not in the solution, there would be no change to the OFV.

Supplier 3 The range for the coefficient of X_3 is an increase of 0.828571429 and a decrease of 2.327272727 from its current value of 5.1. Since $X_3 = 1.3$, an increase of 30 cents (i.e. 0.30) would increase the OFV by 0.30(1.3) = \$0.39. An increase of 85 cents would be beyond the range; the OFV would increase by at least $0.828571429(1.3) \approx \$1.077$. A decrease by \$2 would cause the OFV to fall by \$2(1.3) = \$2.60.

Supplier 4 The current cost for purchases from Supplier 4 is \$2.90 per kg; this has an allowable increase of \$1.706666667 and an allowable decrease of \$0.773333333. Hence a decrease of 50 cents is within the range, a decrease of a dollar would be outside the range, and an increase of \$1.50 would be within the range. Since $X_4 = 0.6$, a 50 cent decrease would cause the OFV to fall by $\$0.50(0.6) = \$0.30$, a one dollar decrease would cause the OFV to fall by at least $\$0.773333333(0.6) = \$0.464$, and \$1.50 increase would cause the OFV to rise by $\$1.50(0.6) = \$0.90$.

Supplier 5 They pay \$3.10 per kg from Supplier 5 and are currently ordering 7.8 kg per bag of final product. The allowable increase is \$1.866666667 and the allowable decrease is \$8.533333333. Hence a decrease of \$5 would be within the allowable range but an increase of \$5 would be outside the range. A decrease of \$2 would cause the OFV to fall by $\$2.00(7.8) = \$15.60$. An increase of \$5 would cause the OFV to rise by at least $\$1.866666667(7.8) = \$14.56$.

Changes to the Right-Hand-Side Values

We consider what happens to the OFV in each of the following situations:

1. The right-hand side value (rhs) of the Nutrient 1 constraint: (a) increases by 1.5; (b) decreases by 1.6; (c) decreases by 2.1.
2. The rhs of the Nutrient 2 constraint: (a) increases by 3.0; (b) decreases by 15; (c) decreases by 20.
3. The rhs of the Nutrient 3 constraint: (a) increases by 1.5; (b) increases by 2.5.
4. The rhs of the fat constraint: (a) decreases by 0.5; (b) decreases by 1.8; (c) increases by 0.34.
5. The rhs of the mass constraint: (a) decreases by 0.9 kg; (b) increases by 300 g; (c) increases by 700 g.

Nutrient 1 The rhs of the Nutrient 1 constraint is currently 40.5. This binding constraint has an allowable increase of 1.8 and an allowable decrease of 1.71. The shadow price is 1.088888889. An increase of 1.5 would cause the OFV to increase by $1.088888889(1.5) \approx \$1.633$. A decrease of 1.6 is allowable, so the change to the OFV would be $1.088888889(-1.6) \approx \$-1.742$, i.e. $\Delta\,\mathrm{OFV} = -\$1.742$. Notice that decreasing the rhs of this $\geq$ constraint makes the restriction less stringent, and the cost decreases as a result. A decrease of 2.1 is beyond the allowable range; the OFV would change by at least $1.08888889(-1.71) = \$-1.862$, or we could say that it will decrease by at least \$1.862.

Nutrient 2 For the Nutrient 2 constraint, the allowable range is $-16.92 \leq \Delta\,\mathrm{rhs} \leq 3.6$, and the shadow price is \$0.311111111. Hence an increase of 3 would be allowable, as would a decrease of 15, but a decrease of 20 would be beyond the allowable range. An increase of 3 would cause an increase of $(\$0.311111111)3 \approx \$0.933$. A decrease of 15 would cause a change of $(\$0.311111)(-15) \approx -\$4.667$, i.e. the OFV would decrease by \$4.667. A decrease of 20 would cause a change of at least $(\$0.311111)\,(-16.92) = -\$5.264$, i.e. the OFV would decrease by at least \$5.264.

Nutrient 3 The Nutrient 3 constraint is non-binding, which makes things easy. The allowable increase is 1.9, hence an increase of 1.5 would have no effect at all, while an increase of 2.5 would cause there to be a new solution. Because such a change would reduce the feasible region, it would cause the OFV to be impaired (i.e. rise in this situation).

Fat The fat constraint has an allowable increase of 2.127272727, an allowable decrease of 0.72, and a shadow price of -1.422222222. A decrease of 0.5 is therefore allowable, and would cause the OFV to change by $-1.422222(-0.5) \approx \$0.711$, i.e. Δ OFV $= \$0.711$. A decrease of 1.8 is beyond the allowable range; the OFV would change by at least $-1.422222(-0.72) = \$1.024$, i.e. Δ OFV ≥ 1.024. An increase of 0.34 is allowable, and would cause the OFV to change by $-1.422222222(0.34) \approx \$0.484$, i.e. Δ OFV $= -\$0.484$.

Mass Finally the mass constraint has an allowable increase of 0.327272727, an allowable decrease of 1.017391304, and a shadow price of 5.055555556. The current rhs value is 14.4, and the units are kg (kilograms). A decrease of 0.9 kg is therefore allowable, and the change to the OFV would be $5.055555(-0.9) = -\$4.55$. In other words, the OFV would fall by \$4.55. Occasionally a conversion factor is required to analyze something; we re-state the 300 grams as 0.3 kg for consistency with the way the constraint was written. An increase of 0.3 kg is allowable, and the change to the OFV is $(5.055555556)0.3 \approx \$1.5174$, i.e. Δ OFV $= \$1.517$. An increase of 700 g or 0.7 kg exceeds the allowable increase, the OFV would rise by at least $5.055555(0.327273) = \$1.655$.

4.4 Two or More Changes

When two or more (rather than one) coefficients are varied, a new level of complexity is introduced. Ironically, the effect of changing *all* the c_j's or all the b_i's may be easier to analyze than only changing some of them, so we begin with these special cases.

4.4.1 Two Special Cases

If the objective function is changed by multiplying each coefficient by the same positive number, then the optimal solution is unaffected, except that OFV* is also multiplied by the same positive number. For example

$$\min\ 5X_1 + 8X_2 + 7X_3$$

could be changed to

$$\min\ 10X_1 + 16X_2 + 14X_3$$

without affecting the optimal values of the variables. The OFV of the second function will be twice that of the first. One way of understanding this property is by thinking of the first objective function being in pounds sterling, and the second being in dollars, with the exchange rate being £ = \$2.00. Another way of understanding this property is to think of making an isovalue line in the graphical method. As long as one objective function is merely a positive multiple of another, their isovalue lines will be parallel and are optimized at the same corner of the feasible region.

Another straightforward case is when each right hand side value is multiplied by the same positive (> 0) constant. In this situation the solution will change, but it is easy to predict how it will change. If the initial model has right hand side values $b_1, \ldots, b_m$, and an optimal solution $X_1^*, \ldots, X_n^*$, and if the new model is the same except that the right hand sides are now $kb_1, \ldots, kb_m$, where $k > 0$, then the new optimal solution will be $kX_1^*, \ldots, kX_n^*$. The new OFV will be k times the initial OFV.[6]

4.4.2 General Case (Based on the Answer and Sensitivity Reports)

Here we are interested in obtaining information about two or more changes using only the Answer and Sensitivity Reports from the Excel Solver.

When two (or more) simultaneous changes are made to either the objective function coefficients or the right hand side values of a linear optimization model, there are four known situations which do not require the running of a new model:

---

[6]This property can be seen by defining $Y_j = \frac{X_j}{k}$ for $j = 1, \ldots, n$. Hence constraint i of the new model can be written successively as:

$$\sum_{j=1}^{n} a_{ij} X_j = kb_i$$

therefore

$$\sum_{j=1}^{n} a_{ij} kY_j = kb_i$$

therefore

$$\sum_{j=1}^{n} a_{ij} Y_j = b_i$$

Because this has a form identical with the original model, $Y_j^* = X_j^*$ for each value of j. Hence the *new* value of X_j^* is k times its former value.

1. Changing the objective function coefficients of two (or more) variables which are not in the solution (i.e. the values of the variables are 0). If both (or all) the proposed changes to the c_j coefficients are within the allowed ranges, then the current solution remains optimal. The OFV does not change, because the non-solution variables contribute nothing to it.

2. Changing the right hand side coefficients of two (or more) *non-binding* constraints. If both (or all) the proposed changes in the b_i coefficients are within the allowed ranges, then the current solution remains unchanged. Since the solution does not change, the OFV remains the same.

3. Changing the objective function coefficients of two (or more) variables, at least one of which is in the solution (i.e. the value of the variable is $>$ 0). In this case we need to use the "100% Rule" for objective function coefficients.[7] Suppose that we wish to change the coefficient of variable X_j by an amount Δc_j, which is in the allowable range.

   If $\Delta c_j > 0$, we define

   $$r_j = \frac{\Delta c_j}{\text{allowable increase in } c_j}$$

   If $\Delta c_j < 0$, we define

   $$r_j = \frac{|\Delta c_j|}{\text{allowable decrease in } c_j}$$

   If $\Delta c_j = 0$, then $r_j = 0$.

   Hence for all j, $0 \le r_j \le 1$. The 100% Rule is that if $\sum_{j=1}^{n} r_j \le 1$, then the current solution will remain optimal. (The "100%" comes from the fact that 1, the rhs of the equation, is 100%.) If the condition does not hold, i.e. if $\sum_{j=1}^{n} r_j > 1$, then we cannot conclude anything one way or the other.

4. Changing the right hand side values of two (or more) constraints, where at least one of these is binding. In this case we need to use the "100% Rule" for right hand side coefficients. Suppose that we wish to change the right hand side of constraint i by an amount Δb_i, which is in the allowable range.

[7]The 100% rules are due to Bradley, Stephen P., Arnoldo C. Hax, and Thomas L. Magnanti, *Applied Mathematical Programming* (Reading, MA: Addison-Wesley, 1977).

If $\Delta b_i > 0$, we define

$$r_i = \frac{\Delta b_i}{\text{allowable increase in } b_i}$$

If $\Delta b_i < 0$, we define

$$r_i = \frac{|\Delta b_i|}{\text{allowable decrease in } b_i}$$

If $\Delta b_i = 0$, then $r_i = 0$.

Hence for all i, $0 \leq r_i \leq 1$. As before, the 100% Rule is that if $\sum_{i=1}^{m} r_i \leq 1$, then the set of binding constraints remain unchanged. The position of the optimal corner will shift, just as in the case of a single change to the right hand side, but it is still the same corner. If the condition holds, the shadow prices are unaffected. If the condition does not hold, i.e. if $\sum_{i=1}^{m} r_i > 1$, then we cannot conclude anything one way or the other.

4.4.3 Using the 100% Rules – An Example

Here we illustrate the use of the 100% rules, using the Wood Products example (introduced on page 154). Here is the Sensitivity Report (first seen on page 173).

Variable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
$B$5	Spindles/Hour Type 1	7.5	0	3	1	1.4
$C$5	Spindles/Hour Type 2	6	0	4	3.5	1

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
$D$8	Cutting	135	0.111111111	135	54	13.5
$D$9	Polishing	54	0.583333333	54	3.6	18
$D$10	Varnishing	48.75	0	58.5	1E+30	9.75
$D$11	Painting	28.5	0	57	1E+30	28.5

Changes to the c_j Coefficients

In this model, if we were to decrease c_1 (the objective function coefficient of X_1, the number of Type 1 spindles made each hour) to 2.3 and increase c_2 to 5.6, would the current solution remain optimal?

The relevant data comes from the Sensitivity Report. The current value of c_1 is 3, and the allowable decrease is 1.4; the current value of c_2 is 4 and the allowable increase is 3.5.

Hence

$$r_1 = \frac{|2.3 - 3|}{1.4} = 0.5$$

and

$$r_2 = \frac{5.6 - 4}{3.5} = 0.457143$$

Hence $r_1 + r_2 = 0.957143 \leq 1.0$. The condition holds and therefore the current solution remains optimal: X_1^* remains at 7.5, and X_2^* remains at 6.0. The OFV changes of course; its new value is $2.3X_1^* + 5.6X_2^* = 50.85$.

Figure 4.5 shows the region where the 100% rule for objective function coefficients applies. This region is the quadrilateral ACBD, which is shaded in gold. This is a segment of the region shaded in gold and blue of infinite size in which the two coefficients result in the same optimal solution. When for a particular combination (c_1, c_2) the 100% rule shows that the left-hand side is ≤ 1, this corresponds with a point in the gold region. When the left-hand side is > 1, the point is either in the blue region or the white region. Therefore, based on the 100% rule alone, we cannot conclude anything one way or the other. This is why the 100% rule is a sufficiency condition, but not a necessary condition. We see that the graphical analysis done earlier in this chapter yields far more information than does the 100% rule. However, the graphical analysis can only be done when there are just two variables, while the 100% rules work on models of any size.

Changes to the b_i Coefficients

Let us consider the following changes to the rhs values of the Wood Products model:

(1)	Cutting	from 135 to 163	$(\Delta b_1 = 28)$
(2)	Polishing	from 54 to 48	$(\Delta b_2 = -6)$
(3)	Varnishing	from 58.5 to 59	$(\Delta b_3 = 0.5)$
(4)	Painting	from 57 to 54	$(\Delta b_4 = -3)$

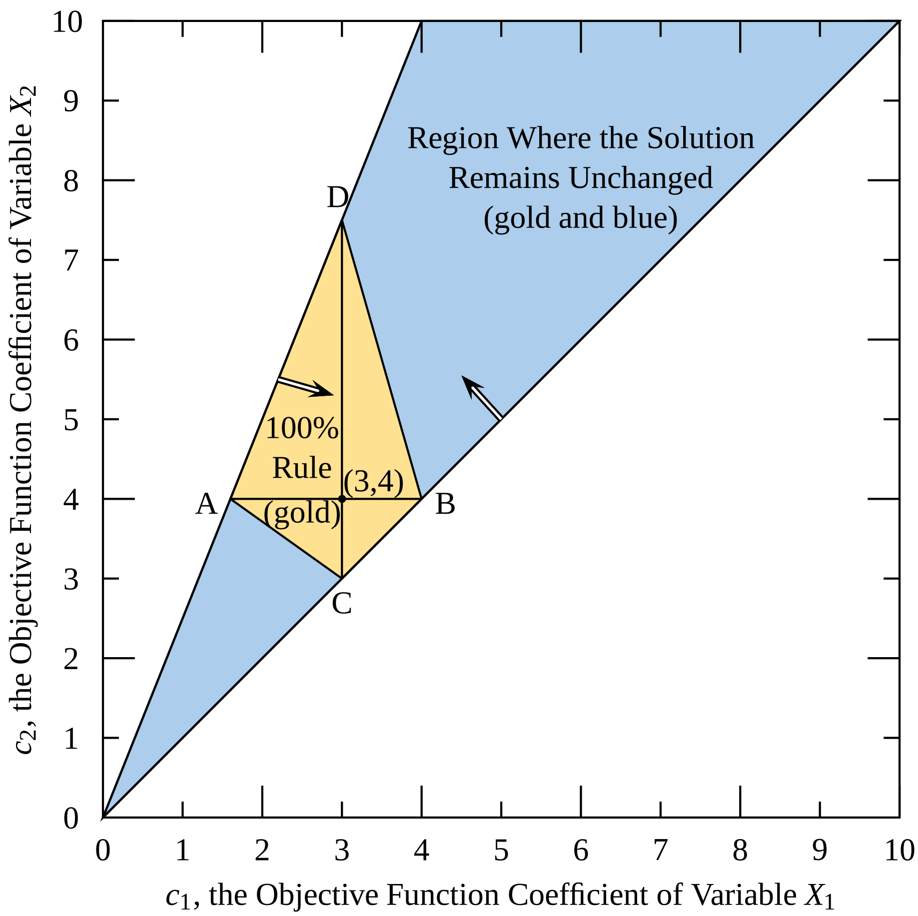


Figure 4.5: Region of (c_1, c_2) where the 100% Rule Applies

Does the current set of binding constraints remain optimal?

The changes to the rhs of both the cutting and varnishing constraints are proposed increases, so from the Sensitivity report we find that the allowable increase for the cutting constraint is 54, and for the varnishing constraint it is infinite. Both the polishing and painting constraints have proposed decreases, so from the Sensitivity report we find that the allowable decrease for the polishing constraint is 18, and for the painting constraint it is 28.5.

Hence we calculate the r_i's as:

$$\begin{array}{rcccl} r_1 & = & \frac{28}{54} & = & 0.5185 \\ r_2 & = & \frac{6}{18} & = & 0.3333 \\ r_3 & = & \frac{0.5}{\infty} & = & 0.0000 \\ r_4 & = & \frac{3}{28.5} & = & 0.1053 \\ \hline \sum_{i=1}^{4} r_i & & & = & 0.9571 \end{array}$$

Since $0.9571 \leq 1$ the condition is met and therefore constraints (1) and (2) are still binding. We can therefore find the new values of X_1^* and X_2^* by solving two equations in two unknowns, which come from the two binding constraints with their new right-hand side values.

$$\begin{array}{rcrcr} 6X_1 & + & 15X_2 & = & 163 \\ 4X_1 & + & 4X_2 & = & 48 \end{array}$$

Solving we obtain $X_1^* = 1\frac{8}{9}$ and $X_2^* = 10\frac{1}{9}$. The new OFV is $3 \times 1\frac{8}{9} + 4 \times 10\frac{1}{9} = 46\frac{1}{9}$ or about \$46.1111.

However, if all we wish to obtain is the change to the OFV, we just need to use the two shadow prices and the changes to the right-hand side values. The shadow prices are $\frac{1}{9}$ for the cutting constraint and $\frac{7}{12}$ for the polishing constraint. Hence $\Delta\text{OFV} = 28 \times \frac{1}{9} + (-6) \times \frac{7}{12} = -0.388888$. If we add this to the current value of \$46.50, we obtain \$46.1111.

4.5 Summary

Managers often need to know how the optimal solution to a model might change if one or more of the parameters of the model were to change. By a graphical analysis for two-variable models, or by using the Excel Solver for larger models, one can identify a range for a particular c_j coefficient for which the optimal solution will not change, or a range for a particular b_i value for which the optimal set of binding constraints will not change. Sometimes, we are able to determine what happens when two or more coefficients are altered using only the final Excel Solver output.

4.6 Problems for Student Completion

4.6.1 Sensitivity Analysis by Graphing

$$\begin{array}{rrcrcr}
\text{maximize} & 7X_1 & + & 5X_2 & & \\
\text{subject to} & & & & & \\
(1) & 4X_1 & + & 6X_2 & \leq & 24 \\
(2) & & & 2X_2 & \leq & 7 \\
(3) & 8X_1 & + & 4X_2 & \leq & 32 \\
(4) & 12X_1 & + & 10X_2 & \leq & 60 \\
& & & & & \\
& X_1 & , & X_2 & \geq & 0
\end{array}$$

(a) Solve the model above graphically.

(b) Suppose that the objective function is now maximize $c_1X_1 + c_2X_2$. Perform a sensitivity analysis to determine when the current solution remains optimal in the following cases:

  (i) both c_1 and c_2 may vary;

  (ii) $c_2 = 5$, c_1 may vary;

  (iii) $c_1 = 7$, c_2 may vary.

(c) Perform a sensitivity analysis for the non-binding constraints.

(d) Perform a sensitivity analysis for the binding constraints, finding the allowable range for each constraint and determine the algebraic expressions for the variables.

(e) From the last part of (d), find the shadow price for each binding constraint.

4.6.2 A Maximization Problem

A garment factory can make skirts, blouses, and dresses. After deducting all variable costs, the net revenue is \$32 per skirt, \$27 per blouse, and \$40 per dress. There are three operations, each of which limits the amount of production: cutting, assembly, and finishing. In addition, each garment must be inspected. Since union rules require that at least one inspector be on duty at all times, they will make a constraint to keep at least one inspector busy. The model has been formulated as:

Let X_1, X_2, and X_3 represent respectively the number of skirts, blouses, and dresses to be made each hour.

$$
\begin{array}{rrcrcrcr}
\text{maximize} & 32X_1 & + & 27X_2 & + & 40X_3 & & \\
\text{subject to} & & & & & & & \\
\text{Cutting} & 5X_1 & + & 4X_2 & + & 2X_3 & \leq & 64 \\
\text{Assembly} & 12X_1 & + & 6X_2 & + & 8X_3 & \leq & 160 \\
\text{Finishing} & 7X_1 & + & 5X_2 & + & 8X_3 & \leq & 146 \\
\text{Inspection} & 6X_1 & + & 4X_2 & + & 3X_3 & \geq & 72 \\
\text{non-negativity} & X_1 & , & X_2 & , & X_3 & \geq & 0
\end{array}
$$

(a) Solve using a spreadsheet Solver, and create the Answer and Sensitivity Reports.

(b) State the solution in words, and indicate which constraints are binding.

(c) By using the information from the Sensitivity Report (rather than by re-running the model each time), give the predicted change to the objective function value (and the reasoning behind your answer) for the following situations (where each situation is independent of the others). If the OFV cannot be predicted exactly, then give an answer such as "the OFV will increase by at least \$90".

(i) The price of each skirt rises by \$5.00.

(ii) There are three fewer units of assembly.

(iii) The price of each dress falls from \$40 to \$27.

(iv) The number of units of cutting increases by 10.

(v) The number of units of finishing increases by 6.

(vi) The price of a blouse increases by \$1.50, and the price of a dress increases by \$2.00.

(vii) The price of a blouse decreases by \$1.20, and the price of a dress increases by \$1.40.

(viii) There are now ten more units of cutting, but one fewer unit of finishing.

(ix) Six more units of cutting become available, and there are now two more units of finishing.

4.6.3 A Minimization Problem

A company which makes chocolate bars needs to buy some exotic nuts: walnuts, chestnuts, and hazelnuts. They do not have to buy any of any one type, but they do need to satisfy certain combinations of types, which has been modelled using three constraints. Also, there is a capacity restriction. The model has been formulated as:

Let X_1, X_2, and X_3 represent respectively the number of kilograms of walnuts, chestnuts, and hazelnuts to be used each hour in the chocolate bar plant.

$$
\begin{array}{rrcrcrcr}
\text{minimize} & 2X_1 & + & 7X_2 & + & 4X_3 & & \\
\text{subject to} & & & & & & & \\
\text{Combination 1} & 5X_1 & + & 8X_2 & + & 6X_3 & \geq & 230 \\
\text{Combination 2} & 2X_1 & + & X_2 & + & 4X_3 & \geq & 145 \\
\text{Combination 3} & 3X_1 & + & 4X_2 & + & 5X_3 & \geq & 196 \\
\text{Capacity} & 8X_1 & + & 9X_2 & + & 4X_3 & \leq & 252 \\
\text{non-negativity} & X_1 & , & X_2 & , & X_3 & \geq & 0
\end{array}
$$

(a) Solve using a spreadsheet Solver, and create the Answer and Sensitivity Reports.

(b) State the solution in words, and indicate which constraints are binding.

(c) By using the information from the Sensitivity Report (NOT by re-running the model each time), give the predicted change to the objective function value (and the reasoning behind your answer) for the following situations (where each situation is independent of the others). If the OFV cannot be predicted exactly, then give an answer such as “the OFV will decrease by at least \$50”.

(i) The price of hazelnuts rises by \$1.20 per kg.

(ii) The price of chestnuts falls by \$2.70 per kg.

(iii) An extra 100 units of capacity becomes available.

(iv) The requirement for combination 1 falls by 25 units.

(v) The requirement for combination 3 increases by 92 units.

(vi) The price of walnuts rises by 15 cents per kg, while the price of hazelnuts falls by 40 cents per kg.

(vii) The price of chestnuts falls by \$4 per kg, while the price of hazelnuts rises by \$1 per kg.

(viii) The right-hand side value of Combination 3 rises by 80 units, while the RHS of Capacity rises by 20 units.

(ix) The RHS of Combination 3 decreases by 4 units, while the RHS of Capacity increases by 21 units.

4.6.4 Parametric Analysis

Sometimes we wish to analyze the effect of changing a parameter over a wide range of values. Performing changes over a wide range is known as *parametric analysis*. This can be accomplished by using the sensitivity analysis to establish the range above and below the current value, and then changing the current value to a number outside the current range to find a new range for this parameter. For example, consider the following model:

$$
\begin{array}{rrcrcr}
\text{minimize} & 5X_1 & + & 8X_2 & & \\
\text{subject to} & & & & & \\
(1) & 2X_1 & + & 5X_2 & \geq & 910 \\
(2) & 4X_1 & + & 3X_2 & \geq & 1092 \\
(3) & X_1 & + & 9X_2 & \geq & 819 \\
 & & & & & \\
 & X_1 & , & X_2 & \geq & 0
\end{array}
$$

(a) Solve this model graphically.

(b) From the graph, perform a sensitivity analysis on b_2, the rhs value of constraint (2).

(c) Re-solve the model using the Excel Solver.

(d) Now we consider changes to b_2 beyond what we determined in part (b). The set of constraints which bound the feasible region changes several times as b_2 is varied, but between these changes the shadow price within a specific allowable range will be constant. Use Excel to determine the set of allowable ranges for $0 \leq b_2 \leq 4000$.

(e) Make a graph of the optimal OFV as a function of b_2, for $0 \leq b_2 \leq 4000$.

Chapter 5

Network Models

In this chapter we consider several types of "network" models. The word *network* comes from the fact that all these models can be thought of as connected points on a physical network. That being said, in the original formulation of these models, each was developed on its own, and it is only later that they came to be studied as a particular class of models. All of these models have their own specialized algorithms (i.e. a procedure for finding the solution). However, to solve models for the assignment, transportation, and transshipment problems, we will not use purpose-built algorithms, but instead will use the Excel Solver (for which the underlying algorithm is the simplex algorithm). We will also study the minimum spanning tree problem (which has a very easy visual algorithm for its solution), the maximum flow problem, and the shortest path problem. The two latter problems will be solved by the Excel Solver.

5.1 Assignment Problem

First, we present a small example of this type of problem, and then we will examine the general model.

5.1.1 Example: Assigning 3 Jobs to 3 Machines

Suppose that we have three jobs, and three machines on which these jobs will be done. Each machine will do just one of the three jobs. All three machines are capable of doing each job, but there are some differences in performance. We can think of these differences in terms of cost (which could be time rather than dollars). Suppose that the costs (in tens of dollars) to assign each job (row) to each machine (column) are as follows:

		Machine		
		1	2	3
	1	30	20	18
Job	2	17	40	21
	3	25	32	28

If we merely wished the numerical answer to this little example, we could easily find it by inspection. The minimum cost solution is to assign job 1 to machine 2, assign job 2 to machine 1, and assign job 3 to machine 3, for a total cost of $20+17+28 = 65$ units of tens of dollars, i.e. \$650. However, finding a solution by inspection won't be possible if we are trying to assign ten jobs to ten machines. Therefore, we will build an algebraic model for the example, and this will help us create a general algebraic model for a problem of any size.

It might be tempting the think of this problem as needing only three variables, with each representing the machine number to which each of the three jobs should be assigned. However, this approach doesn't help us. Instead, we need to think of each pair of job and machine. Should job 1 be assigned to machine 1? Should job 1 be assigned to machine 2? Continuing in this manner, we obtain nine (three times three) "yes or no" type questions. This leads us to formulate this model with nine variables. For each, the "yes" or "no" is modeled with the numbers 1 and 0, respectively. It is useful in a problem like this to have double-subscription on the variable names. The first subscript number indicates the job, and the second the machine. For example, $X_{1,3}$ is used for the pair (job 1, machine 3). For this variable, the binary choice is:

$$X_{1,3} = \left\{ \begin{array}{ll} 1 & \text{if job 1 is assigned to machine 3} \\ 0 & \text{otherwise} \end{array} \right\}$$

It would be tedious to write out all nine variables this way. Instead of defining each variable separately, the variable definitions can be written in one expression, where we define the meaning of X_{ij} for all pairs (i, j):

$$X_{ij} = \left\{ \begin{array}{ll} 1 & \text{if job } i \text{ is assigned to machine } j \\ 0 & \text{otherwise} \end{array} \right\} \qquad i = 1,2,3 \qquad j = 1,2,3$$

The reason for using the numbers 1 and 0 becomes clear when we write the model. For example, if job 2 is assigned to machine 3 (i.e. $X_{2,3} = 1$), then the cost is 21(1) = 21. If job 2 is *not* assigned to machine 3 (i.e. $X_{2,3} = 0$), then the cost is 21(0) = 0. Hence, whether or not job 2 is assigned to machine 3, we incur a cost of $21X_{2,3}$.

Hence the objective function is:

$$\text{minimize } 30X_{1,1}+20X_{1,2}+18X_{1,3}+17X_{2,1}+40X_{2,2}+21X_{2,3}+25X_{3,1}+32X_{3,2}+28X_{3,3}$$

Every job must be assigned to a machine, hence for each job i one of X_{ij}'s will be 1 (and the other two will be 0), hence the sum will be 1:

$$\begin{aligned} X_{1,1}+X_{1,2}+X_{1,3} &= 1 \\ X_{2,1}+X_{2,2}+X_{2,3} &= 1 \\ X_{3,1}+X_{3,2}+X_{3,3} &= 1 \end{aligned}$$

Every machine must have a job assigned to it, hence for each machine j one of X_{ij}'s will be 1 (and the other two will be 0), hence the sum will be 1:

$$\begin{aligned} X_{1,1}+X_{2,1}+X_{3,1} &= 1 \\ X_{1,2}+X_{2,2}+X_{3,2} &= 1 \\ X_{1,3}+X_{2,3}+X_{3,3} &= 1 \end{aligned}$$

Finally, the model ends not with the usual non-negativity restrictions, but instead the fact that each variable must be 0 or 1 is noted. One way to write this is:

$$\text{all } X_{ij} \in \{0,1\}\,.$$

In one sense this is a specialized type of linear programming problem, but it seems to violate one of the assumptions of linear programming which requires that all variables be continuous, rather than integer. However, it turns out that the assignment problem is naturally integer. By this, we mean that the solution will only contain 0/1 variables, even when these have not been specifically required. Hence, any software for general linear programming will solve an assignment problem.

The special structure of the formulation (all left-hand side coefficients are either 0 or 1) has enabled researchers to find dedicated algorithms for the assignment problem, which are computationally much more efficient than the simplex algorithm. The study of such algorithms is beyond the scope of this chapter. We will use the Solver to solve this type of problem. Indeed, the rectangular array paradigm of Excel is very useful for this type of problem, where the cost data is in this format in the first place.

Since the cost data are in a 3 by 3 array, we can also use a 3 by 3 array for the values of the variables. Note that the SUMPRODUCT function is happy with this; here it's an array times an array on a cell-by-cell basis, not the dot product of one row with another row.[1] Here is the setup in formula mode on the spreadsheet, before entering the Solver:

	A	B	C	D	E	F	G
1	**Assignment**		Machine				
2	**Problem**	1	2	3	sum		
3	Job 1				=SUM(B3:D3)	=	1
4	Job 2				=SUM(B4:D4)	=	1
5	Job 3				=SUM(B5:D5)	=	1
6	sum	=SUM(B3:B5)	=SUM(C3:C5)	=SUM(D3:D5)			
7		=	=	=			
8		1	1	1			
9							
10	Total Cost	30	20	18			
11	=SUMPRODUCT(B3:D5,B10:D12)	17	40	21			
12		25	32	28			

In the Solver we ask it to minimize A11 by changing variable cells B3:D5, subject to the three constraints E3:E5 = G3:G5, and the three constraints B6:D6 = B8:D8. We click on the "Make unconstrained variables non-negative" box, and ask for the problem to be solved using the "Simplex LP". Solving the model we obtain:

[1]In this example it's B3 times B10 plus C3 times C10 and so on up to D3 times D12. This type of product is not the same as matrix multiplication.

	A	B	C	D	E	F	G
1	**Assignment**		Machine				
2	**Problem**	1	2	3	sum		
3	Job 1	0	1	0	1	=	1
4	Job 2	1	0	0	1	=	1
5	Job 3	0	0	1	1	=	1
6	sum	1	1	1			
7		=	=	=			
8		1	1	1			
9							
10	Total Cost	30	20	18			
11	65	17	40	21			
12		25	32	28			

As we saw earlier, we see from the Solver output that the minimum cost solution is to assign job 1 to machine 2, assign job 2 to machine 1, and job 3 to machine 3, with a total cost of 65 units, i.e. $650.

5.1.2 Assigning n Jobs to n Machines

Now we consider the general assignment problem, in which there are n jobs to be assigned to n machines, such that each machine does exactly one job, and the cost of assigning job i to machine j is c_{ij}. We define:

$$X_{ij} = \left\{ \begin{array}{ll} 1 & \text{if job } i \text{ is assigned to machine } j \\ 0 & \text{otherwise} \end{array} \right\} \qquad i = 1, 2, \ldots, n \quad j = 1, 2, \ldots, n$$

The objective function is:

$$\text{minimize} \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} X_{ij}$$

Each job must be assigned to a machine, therefore we need the following n constraints:

$$\text{job } i \text{ assigned to a machine} \qquad \sum_{j=1}^{n} X_{ij} = 1 \quad i = 1, 2, \ldots, n$$

Each machine must have a job assigned to it, therefore we need the following n constraints:

$$\text{machine } j \text{ is assigned a job} \quad \sum_{i=1}^{n} X_{ij} = 1 \quad j = 1, 2, \ldots, n$$

Finally, we must have:

$$\text{all } X_{ij} \in \{0, 1\}.$$

Just as the size can be generalized, so can the applications. Besides assigning jobs to machines, we could have workers to jobs, trial judges to cases, manuscripts to editors, and so on.

5.1.3 Special Cases

Impossible Assignment

If a particular job cannot be assigned to a particular machine, we could simply add a constraint to disallow this assignment. However, if we want to keep the structure of the model intact, an alternate way to accomplish this is to make the cost coefficient for this pair very high. Aside from an assignment that must be disallowed for technological reasons, we might wish to disallow an assignment for another reason, such as conflict-of-interest. For example, we would not want to assign a trial judge to a case in which his daughter was the accused.

Uneven Situation

Suppose that there are three jobs to be assigned to four machines, hence one machine will not have a job assigned to it. We can handle this in one of the following two ways:

1. We can create a "dummy" job, which would have no cost of being assigned to any machine. Now we assign four jobs (three real ones plus the dummy) to the four machines, and we would then simply ignore the assignment of the dummy. Dedicated algorithms for the assignment problem often assume this "balanced" (i,e. the number of jobs equals the number of machines) case.

2. There is no need for a "dummy" if we are using the Excel Solver, which uses the simplex algorithm. The first set of constraints remain as equality constraints, but the second set simply become $\leq$ constraints.

5.2 Transportation Problem

The *transportation problem* involves sending supplies from origins to satisfy demands at destinations so as to minimize the total cost of shipping. We begin with an example of this problem, and then present the general formulation.

5.2.1 Transportation Example

A large manufacturer of heavy machinery in eastern Canada has three factories located in Toronto, Montréal, and Halifax. Each factory will serve the local market in which it is situated. In addition, each plant has the capacity to produce beyond its local market for markets in five other cities: London, Ottawa, Kingston, Québec City and Fredericton. Since shipments from Toronto, Montréal, and Halifax to the five other cities are made using palettes filled with the company's product, each unit of shipment is a loaded palette. The excess capacities in Toronto, Montréal, and Halifax are 600, 400, and 150 loaded palettes respectively. The requirements at London, Ottawa, Kingston, Québec City and Fredericton are 450, 350, 250, 150 and 100 loaded palettes, respectively.

The general problem is to distribute the required loaded palettes to each of the markets such that the profit can be maximized. If the selling price in each of the areas is the same, then the major profit factor would be the transportation cost. Thus we only need to minimize the cost of transporting the loaded palettes. The following costs (in hundreds of dollars) per loaded palette have been determined for each of the routes.

Source	Destination				
	London	Ottawa	Kingston	Québec City	Fredericton
Toronto	6	11	8	13	17
Montréal	12	9	8	7	10
Halifax	18	13	15	10	5

5.2.2 Model Formulation

An appropriate objective would be to minimize the cost of transporting the loaded palettes from where they are to where they are needed. The only factor under the control of the retailer is the number of loaded palettes to ship from each supply centre to each marketing centre. The factors which constrain the decision makers are the supply limits at each supply centre and the demands at each marketing centre.

In defining our decision variables, it is convenient to use double-subscription notation. The decision variables are (using the word *unit* to mean a *loaded palette*):

$$X_{ij} = \text{number of units shipped from source (supply) } i \text{ to destination (demand) } j$$

where i = 1, 2 and 3 represents Toronto, Montréal and Halifax, and j = 1, 2, 3, 4 and 5 represents London, Ottawa, Kingston, Québec City and Fredericton.

[In a small example like this, it would also be possible to define the variables using a pair of letters: TL represents the number of units shipped from Toronto to London, TO represents the number of units shipped from Toronto to Ottawa, and so on, and finally HF represents the number of units shipped from Halifax to Fredericton.]

The total cost of shipping the units (using the subscripted variables) can be written as:

$$\begin{array}{rcrcrcrcrcr}
\text{OFV} & = & 6X_{1,1} & + & 11X_{1,2} & + & 8X_{1,3} & + & 13X_{1,4} & + & 17X_{1,5} \\
 & + & 12X_{2,1} & + & 9X_{2,2} & + & 8X_{2,3} & + & 7X_{2,4} & + & 10X_{2,5} \\
 & + & 18X_{3,1} & + & 13X_{3,2} & + & 15X_{3,3} & + & 10X_{3,4} & + & 5X_{3,5}
\end{array}$$

We have two different sets of constraints, one associated with the supply restrictions, and the other associated with the demand restrictions.

A. Supply restrictions

$$\begin{array}{rrl}
\text{Toronto} & X_{1,1}+X_{1,2}+X_{1,3}+X_{1,4}+X_{1,5} \leq & 600 \\
\text{Montréal} & X_{2,1}+X_{2,2}+X_{2,3}+X_{2,4}+X_{2,5} \leq & 400 \\
\text{Halifax} & X_{3,1}+X_{3,2}+X_{3,3}+X_{3,4}+X_{3,5} \leq & 350
\end{array}$$

B. Demand restrictions

$$\begin{array}{rrl}
\text{London} & X_{1,1}+X_{2,1}+X_{3,1} \geq & 450 \\
\text{Ottawa} & X_{1,2}+X_{2,2}+X_{3,2} \geq & 350 \\
\text{Kingston} & X_{1,3}+X_{2,3}+X_{3,3} \geq & 250 \\
\text{Québec City} & X_{1,4}+X_{2,4}+X_{3,4} \geq & 150 \\
\text{Fredericton} & X_{1,5}+X_{2,5}+X_{3,5} \geq & 100
\end{array}$$

The total supply is $600+400+350=1350$, and the total demand is $450+350+250+150+100=1300$. Since the total supply meets or exceeds the total demand (in this example the former exceeds the latter by 50 units), the model will have a feasible solution. In summary, the linear optimization model for this transportation problem is:

$$
\begin{array}{lrcrcrcrcrcr}
\text{minimize} & & & 6X_{1,1} & + & 11X_{1,2} & + & 8X_{1,3} & + & 13X_{1,4} & + & 17X_{1,5} \\
& + & & 12X_{2,1} & + & 9X_{2,2} & + & 8X_{2,3} & + & 7X_{2,4} & + & 10X_{2,5} \\
& + & & 18X_{3,1} & + & 13X_{3,2} & + & 15X_{3,3} & + & 10X_{3,4} & + & 5X_{3,5}
\end{array}
$$

subject to

$$
\begin{array}{lrcl}
\text{Toronto} & X_{1,1}+X_{1,2}+X_{1,3}+X_{1,4}+X_{1,5} & \leq & 600 \\
\text{Montréal} & X_{2,1}+X_{2,2}+X_{2,3}+X_{2,4}+X_{2,5} & \leq & 400 \\
\text{Halifax} & X_{3,1}+X_{3,2}+X_{3,3}+X_{3,4}+X_{3,5} & \leq & 350 \\
\text{London} & X_{1,1}+X_{2,1}+X_{3,1} & \geq & 450 \\
\text{Ottawa} & X_{1,2}+X_{2,2}+X_{3,2} & \geq & 350 \\
\text{Kingston} & X_{1,3}+X_{2,3}+X_{3,3} & \geq & 250 \\
\text{Québec City} & X_{1,4}+X_{2,4}+X_{3,4} & \geq & 150 \\
\text{Fredericton} & X_{1,5}+X_{2,5}+X_{3,5} & \geq & 100
\end{array}
$$

$$
\text{non-negativity} \qquad X_{ij} \geq 0 \quad i = 1,3; \quad j = 1,5
$$

5.2.3 General Model

In the general form of the model the parameters of the model are as follows. There are m supply points, and n demand points. The supply at supply point i is s_i, and the demand at demand point j is d_j. In order for there to be a solution, we must have:

$$
\sum_{i=1}^{m} s_i \geq \sum_{j=1}^{n} d_j
$$

We will assume that the s_i's and the d_j's are positive integers. The cost to ship one unit from supply point i to demand point j is c_{ij}.

The unknowns of the model are the quantities to be shipped from each supply point to each demand point. Hence, there are $m \times n$ decision variables, where X_{ij} is the quantity to be shipped from supply point i to demand point j.

The general transportation model is as follows:

$$
\begin{array}{rrcll}
\text{minimize} & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij}X_{ij} & & & \\
\text{subject to} & & & & \\
\text{supplies} & \displaystyle\sum_{j=1}^{n} X_{ij} & \leq & s_i & (i=1,\ldots,m) \\
\text{demands} & \displaystyle\sum_{i=1}^{m} X_{ij} & \geq & d_j & (j=1,\ldots,n) \\
 & X_{ij} & \geq & 0 & \left\{ \begin{array}{c} i=1,\ldots,m \\ j=1,\ldots,n \end{array} \right\}
\end{array}
$$

An important property holds as a result of our assumption that each s_i and each d_j is a positive integer. This property is that each X_{ij} will also be a non-negative integer. The transportation problem is one of the few problems where the integrality of the decision variables occurs in such a natural fashion.[2]

There was a time when all transportation problems had to have balanced supply and demand, which often required the creation of a dummy demand point to absorb the difference between the total demand and the total supply of the original model. This was done because the specialized algorithms which had been written for the transportation problem assumed the balanced situation. However, this is not needed by the simplex algorithm which the Excel Solver uses, hence we will leave all such problems is the original unbalanced form.

5.2.4 Solution

Now we solve the example presented earlier. We use a rectangular array for the cost coefficients (orange), and reserve a rectangular array for the values of the variables (yellow). On Excel we begin with:

[2] In any problem, we can force the variables to be integer by declaring them to be **int** on the Excel Solver. What we are saying here is that in the transportation problem we will obtain integer variables even if we do not make this declaration.

	A	B	C	D	E	F	G	H	I
1		**Transportation Model**							
2		London	Ottawa	Kingston	Quebec C.	Fredericton	sum		
3	Toronto						0	<=	600
4	Montreal						0	<=	400
5	Halifax						0	<=	350
6	sum	0	0	0	0	0			
7		>=	>=	>=	>=	>=			
8		450	350	250	150	100			
9									
10	Total Cost	6	11	8	13	17			
11	0	12	9	8	7	10			
12		18	13	15	10	5			

The following formulas were entered:

1. =SUM(B3:F3) in cell G3, copied to G3:G5.
2. =SUM(B3:B5) in cell B6, copied to B6:F6.
3. =SUMPRODUCT(B3:F5,B10:F12) in cell A11.

In the Solver we:

1. **Set Objective** `A11`.
2. Click on **Min**.
3. Make the **Changing Variable Cells** `B3:F5`.
4. **Subject to the Constraints**
   `G3:G5` $\leq$ `I3:I5` and `B6:F6` $\geq$ `B8:F8`.
5. Click on **Make Unconstrained Variables Non-Negative**.
6. Under **Select a Solving Method** we choose the Simplex LP.

Solving we obtain:

	A	B	C	D	E	F	G	H	I
1		Transportation Model							
2		London	Ottawa	Kingston	Quebec C.	Fredericton	sum		
3	Toronto	450	0	150	0	0	600	<=	600
4	Montreal	0	300	100	0	0	400	<=	400
5	Halifax	0	50	0	150	100	300	<=	350
6	sum	450	350	250	150	100			
7		>=	>=	>=	>=	>=			
8		450	350	250	150	100			
9									
10	Total Cost	6	11	8	13	17			
11	10050	12	9	8	7	10			
12		18	13	15	10	5			

We see that in the optimal solution, from Toronto we send 450 units to London and 150 to Kingston, from Montréal we send 300 units to Ottawa and 100 units to Kingston, and from Halifax we send 50 units to Ottawa, 150 units to Québec City, and 100 units to Fredericton. The unused capacity is 50 units; this occurs at Halifax. The cost of the optimal solution is 10,050 hundreds of dollars, i.e. \$1,005,000.

5.2.5 A Modification to the Example

Suppose that something comes along to change this model. For example, suppose that a fire at their Montréal location has reduced the capacity (in excess of the demand locally) to only 50 units, a drop of 350 units. In response to this catastrophe, the company has decided to increase production in Halifax by operating longer hours. Halifax's capacity is now raised by 300 units from 350 to 650, which combined with the 50 units of unused capacity at Halifax offsets the loss of 350 units at Montréal. The total supply is now 1300 units, which equals the total demand. While this requires a major adjustment for the company, it only requires a minor revision to the model. We simply change the numbers in cells I4 and I5 to 50 and 650, respectively. Doing this and clicking on OK on the Solver we obtain:

	A	B	C	D	E	F	G	H	I
1		**Transportation Model**							
2		London	Ottawa	Kingston	Quebec C.	Fredericton	sum		
3	Toronto	450	0	150	0	0	600	<=	600
4	Montreal	0	0	50	0	0	50	<=	50
5	Halifax	0	350	50	150	100	650	<=	650
6	sum	450	350	250	150	100			
7		>=	>=	>=	>=	>=			
8		450	350	250	150	100			
9									
10	Total Cost	6	11	8	13	17			
11	11600	12	9	8	7	10			
12		18	13	15	10	5			

The total cost has increased to 11,600($100) = $1,160,000. This is to be expected, because now Halifax is serving places such as Ottawa, which is much further away from its new point of supply of Halifax than it was to its previous supply point of Montréal.

5.3 Transshipment Problem

A transshipment problem is an extension of the transportation problem, in which one or more places can be both a supply point and a demand point. Suppose that Montréal can receive supplies from both Toronto and Halifax. All these cities are ports, so it might be possible to move items between these cities cheaply using ships. Let's suppose that there's a $200 per-unit cost between Toronto and Montréal, and a $300 per-unit cost being Halifax and Montréal.

What flows out of Montréal is the same as before, i.e. $X_{2,1} + X_{2,2} + X_{2,3} + X_{2,4} + X_{2,5}$. The flow in is the production capacity at Montréal (400 before the fire, 50 afterwards), plus the amounts received from Toronto and Halifax. We will let TM represent the number of units shipped from Toronto to Montréal, and HM represent the number of units shipped from Halifax to Montréal. In the post-fire situation the flow balance at Montréal is:

$$X_{2,1} + X_{2,2} + X_{2,3} + X_{2,4} + X_{2,5} \leq 50 + TM + HM$$

Hence the revised supply constraints are:[3]

$$\begin{array}{llcr}
\text{Toronto} & X_{1,1}+X_{1,2}+X_{1,3}+X_{1,4}+X_{1,5}+TM & \leq & 600 \\
\text{Montréal} & X_{2,1}+X_{2,2}+X_{2,3}+X_{2,4}+X_{2,5}-TM-HM & \leq & 50 \\
\text{Halifax} & X_{3,1}+X_{3,2}+X_{3,3}+X_{3,4}+X_{3,5}+HM & \leq & 650
\end{array}$$

In modeling this situation in Excel, we need to add Montréal as a destination. We will use the post-fire transportation spreadsheet, and add Montréal as column G. In rows 10 to 12 of column G, the new costs of shipping from Toronto and Halifax to Montréal must be entered. These figures are \$200 and \$300 respectively, but all costs are entered in units of hundreds of dollars, so we put a 2 into cell G10 and a 3 into cell G12. We sum rows 3 to 5 of this column, putting `=SUM(G3:G5)` into cell G6, but there is no constraint on this column. This sum will be $TM+HM$, hence in the new column H, we need to put `=SUM(B4:G4)-G6` into the cell in the Montréal row (i.e. cell H4), which calculates the value of $X_{2,1}+X_{2,2}+X_{2,3}+X_{2,4}+X_{2,5}-(TM+HM)$. In formula mode columns G, H, I, and J are now:

	G	H	I	J
1				
2	Montreal	sum		
3		=SUM(B3:G3)	<=	600
4		=SUM(B4:G4)-G6	<=	50
5		=SUM(B5:G5)	<=	650
6	=SUM(G3:G5)			
7				
8				
9				
10	2			
11	0			
12	3			

In cell A11, we now include column G when calculating the cost:
`=SUMPRODUCT(B3:G5,B10:G12)`.

In the Solver, things are similar to the transportation problem, but we now need to compare columns H and J: `H3:H5` $\leq$ `J3:J5`.

Solving, we obtain:

[3] Since the supply and demand are now balanced, all the constraints could be made equalities, but it's easier just to leave them as $\leq$ supply constraints and $\geq$ demand constraints.

	A	B	C	D	E	F	G	H	I	J
1		**Transshipment Model**								
2		London	Ottawa	Kingston	Quebec C.	Fredericton	Montreal	sum		
3	Toronto	450	0	150	0	0	0	600	<=	600
4	Montreal	0	350	100	0	0	0	50	<=	50
5	Halifax	0	0	0	150	100	400	650	<=	650
6	sum	450	350	250	150	100	400			
7		>=	>=	>=	>=	>=				
8		450	350	250	150	100				
9										
10	Total Cost	6	11	8	13	17	2			
11	11050	12	9	8	7	10	0			
12		18	13	15	10	5	3			

We see that the cost has been reduced from 11,600 to 11,050 hundreds of dollars, i.e. from \$1,160,000 to \$1,105,000.

In this example, transshipment to Montréal was possible from both Toronto and Halifax. Had this not been so, for example if transshipment were possible from Halifax but impossible from Toronto, then we would have needed to put in a large cost coefficient (such as 9999) in the Toronto to Montreal cost cell (cell G10) to prevent any flow from happening.

5.4 Networks

We are all familiar with the notion of a *network* in the sense of a television network. More generally, a network consists of a set of places (called *nodes*) which are connected together. Other examples of networks include:

(i) cities and roads

(ii) oil wells and pipelines

(iii) switching stations and telephone wires.

In this section, we will examine three network problems:

(a) The minimum spanning tree problem: How can the nodes be connected so that the total construction cost is minimized? For this problem, an easy visual algorithm is presented.

(b) The maximum flow problem: In a network with capacity constraints, what is the maximum flow between a given pair of nodes? This problem will be solved using the Excel Solver.

(c) The shortest path problem: What is the shortest (or cheapest, or least time) path through a network between a given pair of nodes? This problem will also be solved using the Excel Solver.

It should be noted that the models seen so far in this chapter for the assignment, transportation, and transshipment problems, are also network models. However, they are more easily understood as applications in their own right instead of thinking of them as specialized networks.

5.4.1 Definition of Terms

In the above examples of networks, the cities, oil wells, switching stations or activities are represented by "nodes," which are drawn as circles or squares, with the node identification number drawn in the middle.

$\boxed{5}$ means "node 5"

The roads, pipelines, telephone wires or events connecting the activities are represented by "arcs," which are straight or curved lines drawn between pairs of nodes. Unidirectional flow (e.g. a one-way street) is shown by an arrowhead; bidirectional flow is shown either by no arrowheads or by arrowheads at each end of the arc.

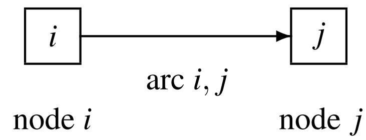
$\longrightarrow$ unidirectional arc

$\longleftrightarrow$ or ——— bidirectional arc

There are two ways of identifying arcs.

1. One way is to give each arc a number which is referenced to a pair of nodes; e.g. a database search shows that arc #152 goes from node 8 to node 14.

2. The other way is to give the arc two numbers, which are the node numbers of the nodes connected by the arc. In the case of a unidirectional arc, the origin node number is given first. The following diagram displays a directed arc from node i to node j.



directed arc i, j

Sometimes, instead of having a bidirectional arc from i to j, there are two unidirectional arcs: one from i to j, and the other from j to i. This is usually done when there is some difference in the two directions. For example, because of winds, it takes more time and fuel to fly a plane from Rome to Toronto than from Toronto to Rome.

5.5 Minimum Spanning Tree Problem

Let us consider a town planning decision associated with building a new subdivision. Storm drains will be located at selected points within the subdivision and we want to connect them to the existing system. We will let each of the storm drains be represented by a node. The location of the drains (the nodes of the network) is given exogenously; our problem is to choose the arcs of the network (drainage pipes in this example) at minimum cost. If node $\boxed{i}$ can be *directly* connected to $\boxed{j}$, then there will be a construction cost c_{ij} for connecting the two drains.

To satisfy the requirements of the town planners, we need to be able to connect the nodes so that it is possible to go from any node to any other node (regardless of how involved the route is), and to have the *total* construction cost minimized. The set of arcs so constructed is referred to as the *minimum spanning tree*.

Other applications are:

(a) to build a road network to connect cities

(b) to build a network of pipelines to connect oil wells

(c) to build a network of cable to connect houses with a cable distribution centre.

Consider the following small example given in Figure 5.1.

There are five potential (bidirectional) arcs which can be used to connect all four nodes. By inspection, the optimal solution (given in Figure 5.2) has a total construction cost of $10 + 7 + 5 = 22$. Note that the solution to this example

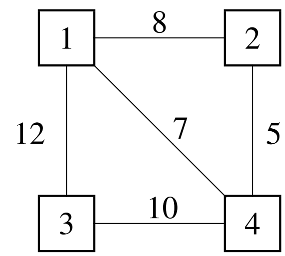


Figure 5.1: A Network With Four Nodes

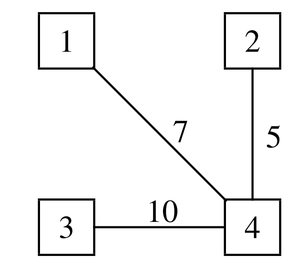


Figure 5.2: Minimum Spanning Tree of the 4-Node Network

with four nodes contains three arcs. It is always true that a network with n nodes network will have $n-1$ arcs in the minimum spanning tree.

Most examples are not as trivial as this one. To solve more complex examples we use an algorithm written especially for finding the minimum spanning tree. We present an example for which the solution is not immediately obvious, and then solve it with a visual algorithm.

5.5.1 An Example With Seven Nodes

Consider the example given in Figure 5.3. The number beside the arc represents what it would cost in hundreds of dollars to construct the link between the beginning and ending nodes of the arc. At the outset none of these costs has occurred – we seek the minimum spanning tree which will require that $7-1=6$ of these links be constructed.

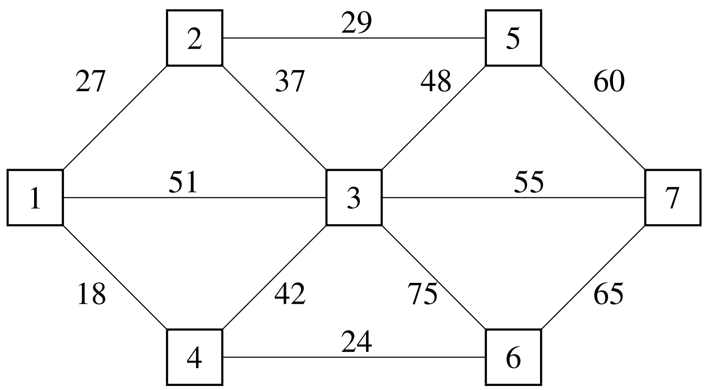
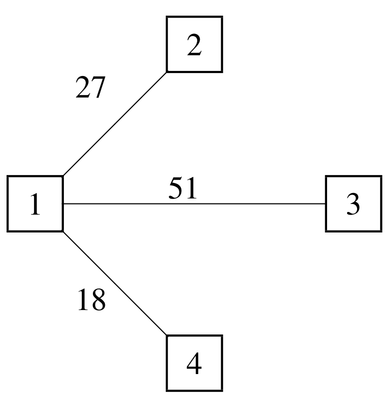


Figure 5.3: A Network with Seven Nodes

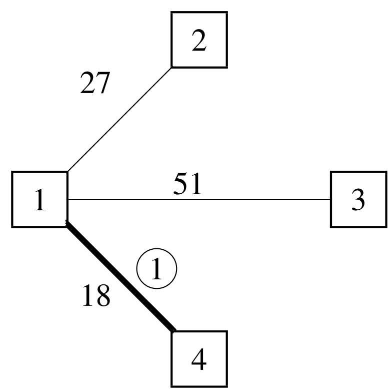
Minimum Spanning Tree – Visual Algorithm

To use this approach we begin with the original network diagram. In practice, all the work is done on this one diagram, though we shall show it with multiple diagrams for pedagogical purposes.

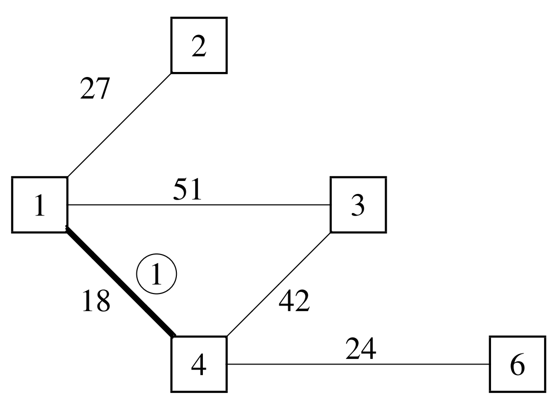
Whatever node we begin with, we will obtain the same solution. Unless otherwise stated, we will begin each application of this algorithm with node 1. This algorithm proceeds myopically – what is amazing is that this myopic approach does indeed obtain the optimal solution. We say that node 1 is *connected*, and that at this moment the other nodes are unconnected. We proceed from this connected node to all unconnected nodes that can be reached directly. In this example, these are nodes 2, 3, and 4. From node 1 we must choose one of the following arcs:



It turns out that the best thing to do is to choose the arc with the lowest number (either representing the least cost, or the least distance). In this example this is arc 1,4 with a cost of 18. We then say that the ending node of this arc (node 4) is *connected*, and this arc enters the solution. To show this, we darken the arc, and show that this was added at iteration 1 by putting a 1 into a circle next to the added arc. The diagram is now:



At the outset of the second iteration, we have everything we have already, plus we show the arcs which can be reached from the newly-added connected node. These are arcs 4,3 and 4,6.



We look at all the arcs which go from connected nodes to unconnected nodes, and choose the cheapest. This is arc 4,6 with a cost of 24. We darken arc 4,6, and show that this was added at the second iteration by placing a circled 2 next to this added arc.

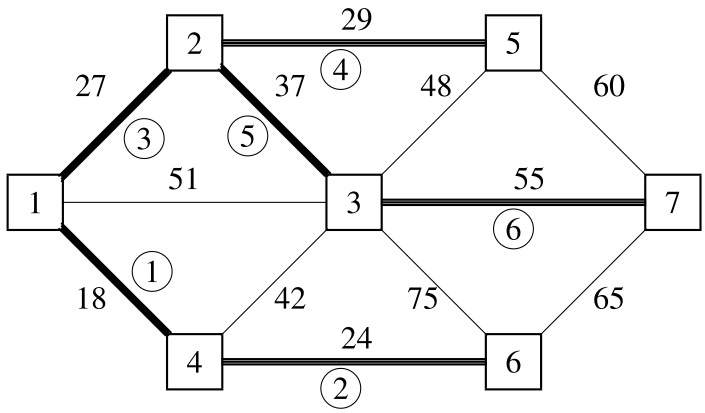
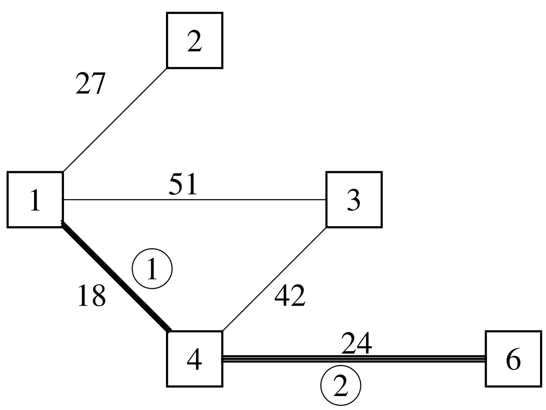


Figure 5.4: 7- Node Minimum Spanning Tree: Visual Solution



At each iteration, the user finds by inspection the least cost arc going from a connected node to an unconnected node. Doing this for four more iterations we obtain what is shown in Figure 5.4.

We see that the total cost is:

$$18 + 24 + 27 + 29 + 37 + 55 = 190$$

Since the units are in hundreds of dollars, the total cost of constructing the links is \$19,000.

The steps of the algorithm can be summarized as:

Step 1: Arbitrarily pick any node and designate that node as being connected to the existing system.

Step 2: For each connected node i which can directly reach unconnected node j, determine the arc with the smallest c_{ij} (break a tie for the smallest c_{ij} arbitrarily). This arc enters the problem solution, and the ending node of this arc is now designated as being connected.

Step 3: If all nodes are connected then STOP. Otherwise, return to Step 2.

Minimum Spanning Tree – Further Comments

In using the preceding algorithm, we are seeing the network diagram. For a computer to obtain the solution, we have store the information in matrix form.

There are three algorithms for the minimum spanning tree problem, which are described at https://en.wikipedia.org/wiki/Minimum_spanning_tree. This article provides links to the three algorithms, and gives an extensive list of references.

Unlike the other network problems in this chapter, we do not provide a way of solving this problem in Excel using the Solver. Aside from this being a slow approach for something which has very fast dedicated algorithms, the algebraic formulation is very difficult. For our purposes, the visual algorithm will suffice.

5.6 The Maximum Flow Problem

A major problem in most large cities is how to manage increasingly heavy traffic flows. Congestion on the road network can cause commuters to spend over an hour to drive to or from work, which gives rise to many related costs. Thus we would like to be able to determine the capacity of an existing network and ways in which it can be expanded most efficiently. A complete analysis here is beyond the scope of this book. Our problem is the maximum flow problem which can be stated as desiring to maximize the flow (e.g. of cars, of cubic metres (m^3) of oil etc) between an origin node ("source") and a destination node ("sink"), subject to capacity constraints on the arcs of the network.

Each arc of the network has a capacity constraint, which might differ according to direction. Suppose we have two pumping stations which we label as nodes 3 and 7. Between them is a pipeline, whose capacity is either 8,000 m^3/day from 3 to 7, or 13,000 m^3/day from 7 to 3.[4] Using units of "thousands of m^3/day" we write the capacity constraints as follows:

[4]We stress that these capacity constraints are either-or, not both.

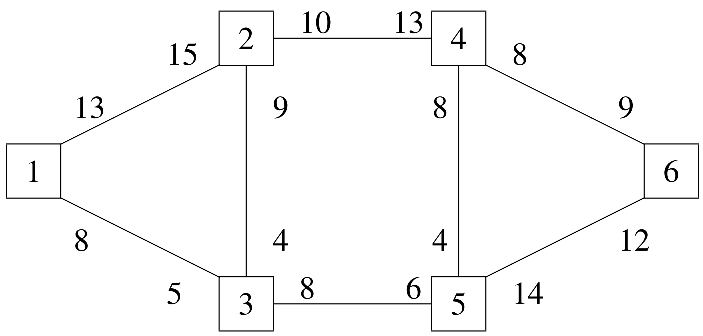
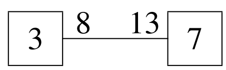


Figure 5.5: Maximal Flow Example – Arc Capacities



A unidirectional arc will have a capacity of 0 in one of the directions.

Consider the example given in Figure 5.5. We wish to know the maximum flow from $\boxed{1}$ to $\boxed{6}$.

5.6.1 The Algebraic Model

We let X_{ij} represent the number of units sent from node i to node j, defined only for those pairs which have an arc between i and j. What we wish to determine is the maximum flow from node 1 to node 6.

To do this, we create a "dummy" arc from node 6 to node 1 which receives flow at 6 and sends it back to 1 over the dummy arc. This flow is $X_{6,1}$. The maximum flow from node 1 to node 6 must equal the return flow over the dummy arc from 6 to 1. Hence, in this example, the objective is:

$$\text{maximize} \quad X_{6,1}$$

One set of constraints comes from the arc capacities. They are simple, but lengthy to write out:

$$
\text{Flow capacities between nodes}\quad
\begin{array}{rcr}
X_{1,2} & \leq & 13 \\
X_{2,1} & \leq & 15 \\
X_{1,3} & \leq & 8 \\
X_{3,1} & \leq & 5 \\
X_{2,3} & \leq & 9 \\
X_{3,2} & \leq & 4 \\
X_{2,4} & \leq & 10 \\
X_{4,2} & \leq & 13 \\
X_{3,5} & \leq & 8 \\
X_{5,3} & \leq & 6 \\
X_{4,5} & \leq & 8 \\
X_{5,4} & \leq & 4 \\
X_{4,6} & \leq & 8 \\
X_{6,4} & \leq & 9 \\
X_{5,6} & \leq & 14 \\
X_{6,5} & \leq & 12
\end{array}
$$

There is no need to write a constraint on the dummy arc in the algebraic model. However, we shall see that it is useful to include such a constraint in an Excel model, and so we might want to include it here too. If used, the upper limit would be set at a figure well beyond whatever the maximum flow could be, for example 1000 units:

$$X_{6,1} \leq 1000$$

The other set of constraints comes from a need for the flow to balance at each node. We must have the Flow In equal to the Flow Out, or equivalently:

$$\text{Flow In} - \text{Flow Out} = 0$$

Note that the flow on the dummy arc ($X_{6,1}$ in this example) is included in the "Flow In" at the beginning node, and is included in the "Flow Out" at the ending node. Hence we need the following six constraints:

$$
\begin{array}{lrcl}
\text{Node 1} & X_{2,1}+X_{3,1}+X_{6,1}-X_{1,2}-X_{1,3} & = & 0 \\
\text{Node 2} & X_{1,2}+X_{3,2}+X_{4,2}-X_{2,1}-X_{2,3}-X_{2,4} & = & 0 \\
\text{Node 3} & X_{1,3}+X_{2,3}+X_{5,3}-X_{3,1}-X_{3,2}-X_{3,5} & = & 0 \\
\text{Node 4} & X_{2,4}+X_{5,4}+X_{6,4}-X_{4,2}-X_{4,5}-X_{4,6} & = & 0 \\
\text{Node 5} & X_{3,5}+X_{4,5}+X_{6,5}-X_{5,3}-X_{5,4}-X_{5,6} & = & 0 \\
\text{Node 6} & X_{4,6}+X_{5,6}-X_{6,4}-X_{6,5}-X_{6,1} & = & 0
\end{array}
$$

Also, we require that all variables be greater than or equal to 0. With all arc capacities being integers, the values of the variables will also be integers.

5.6.2 Excel Model

We could solve the algebraic model in Excel by creating a model with 16 + 1 = 17 columns for each variable cell, one for each arc with non-zero capacity, and one for the dummy arc. However, there is a much easier way to do this.

The arc capacity data was given in a picture (Figure 5.5), but equivalently it could have been given in tabular form. If we have the picture, we can create a table displaying the same information; if we have the information in a table, we can create the picture. Each is an alternate way of stating the same information given in the other form.

From the picture we could make the following table, inserting a 0 capacity for the non-existent arcs, but putting a capacity of 1000 for the dummy arc between nodes 6 and 1.

From \ To	1	2	3	4	5	6
1	0	13	8	0	0	0
2	15	0	9	10	0	0
3	5	4	0	0	8	0
4	0	13	0	0	8	8
5	0	0	6	4	0	14
6	1000	0	0	9	12	0

The reason why we are using an artificially-created capacity for the dummy arc in the Excel model is that it's easier to write the arc capacity constraints this way, as will be demonstrated later. The advantage of using a table is that it's very Excel-friendly; tables use rows and columns, and so does Excel.

Though we don't have $6 \times 6 = 36$ pieces of data, the 6 by 6 array is easier to use than to try to deal with 17 pieces of data separately. When its comes to the variables associated with these 17 pieces of data (capacities), it is again easier to use a 6 by 6 array, but we need to recognize that many of the cells do not represent the defined variables. We therefore reserve a 6 by 6 space in Excel for the variables, but yellow highlighting is only used for the cells representing defined variables.

Here is the initial setup for the Excel model:

	A	B	C	D	E	F	G	H
1		Flows Between Nodes						
2	From \ To	1	2	3	4	5	6	Out
3	1							0
4	2							0
5	3							0
6	4							0
7	5							0
8	6							0
9	In	0	0	0	0	0	0	
10		Capacities Between Nodes						
11	From \ To	1	2	3	4	5	6	Flow
12	1	0	13	8	0	0	0	0
13	2	15	0	9	10	0	0	
14	3	5	4	0	0	8	0	
15	4	0	13	0	0	8	8	
16	5	0	0	6	4	0	14	
17	6	1000	0	0	9	12	0	

In column H, we sum the flows coming out of each node listed in column A, In row 9, we sum the flows going into each node listed in row 2. Though we only need to sum the cells highlighted in yellow, it's easier to sum them all (the other cells will all contain zeroes). Hence in cell H3 we write `=SUM(B3:G3)`, and copy this into the range H3:H8. In cell B9 we write `=SUM(B3:B8)`, and copy this into the range B9:G9.

Though we only wish to maximize $X_{6,1}$, we cannot simply maximize cell B8, as Excel won't allow a variable cell to also be an objective cell. We need to create a dedicated objective cell; we use cell H12 for this purpose, with `=B8` entered into this cell.

Entering the Variable Cells and Constraints on the Solver

There are two approaches for entering the variable cells and the constraints on the Solver. One way is easy; the other way minimizes the amount of computing resources required to solve the problem.

The Easy Way One approach is:

1. Define the entire range B3:G8 as variable cells.

2. Enter the capacity constraints as `B3:G8<=B12:G17`.

3. Enter the node balance constraints as `H3:H8=B9:G9`.

4. Declare all variables to be non-negative. (We do not need to declare the variables to be integer, maximum flow problems are naturally integer.)

Note that one of the constraints entered in operation 2 is for the capacity on the dummy, which is $X_{6,1} \leq 1000$.

Another Way (Optional) In this approach, both items 3 and 4 are as above, but 1 and/or 2 are modified to save on computer resources.

In 1, the "Easy Way" defines 6(6) = 36 variables, but we only need 17. Defining all 36 is easy, because there is only one range to enter, so we might as well do it this way given that this example is small. However, it would be wasteful of computing resources for larger problems. The number of defined variables can be made lower by going to each range of cells highlighted in yellow, and entering each separately. The ranges in this example are:
`C3:D3,B4,D4:E4,B5:C5,F5,C6,F6:G6,D7:E7,G7,B8,E8:F8.`

In 2, the "Easy Way" defines 6(6) = 36 capacity constraints, but we only need 16 of them (the dummy can be omitted). Entering contiguous constraints where possible, we will have to use the "Add Constraint" ten times. These entries are: `C3:D3<=C12:D12`, `B4<=B13`, `D4:E4<=D13:E13`, `B5:C5<=B14:C14`, `F5<=F14`, `C6<=C15`, `F6:G6<=F15:G15`, `D7:E7<=D16:E16`, `G7<=G16`, and `E8:F8<=E17:F17`. Note that with this approach, since no constraint is entered for the dummy in cell B8, it doesn't matter what number we put into cell B17. A variant to this approach would add `B8<=B17`, in which case we would need the 1000 (or other suitable number) in cell B17.

Solution

Solving the model we obtain:

	A	B	C	D	E	F	G	H
1		Flows Between Nodes						
2	From \ To	1	2	3	4	5	6	Out
3	1	0	10	8	0	0	0	18
4	2	0	0	0	10	0	0	10
5	3	0	0	0	0	8	0	8
6	4	0	0	0	0	6	4	10
7	5	0	0	0	0	0	14	14
8	6	18	0	0	0	0	0	18
9	In	18	10	8	10	14	18	
10		Capacities Between Nodes						
11	From \ To	1	2	3	4	5	6	Flow
12	1	0	13	8	0	0	0	18
13	2	15	0	9	10	0	0	
14	3	5	4	0	0	8	0	
15	4	0	13	0	0	8	8	
16	5	0	0	6	4	0	14	
17	6	1000	0	0	9	12	0	

We see that at most 18 units can be shipped from node 1 to node 6. If we wish to find the maximum flow on this network between a different pair of nodes, not much work needs to be done. We would have a new dummy arc replacing the old one, and the changing cells would need to have the new dummy arc cell added, and the old one deleted.

More information about the maximum flow problem can be found at: https://en.wikipedia.org/wiki/Maximum_flow_problem.

5.7 The Shortest Path Problem

There are many situations in which it is important to be able to reach certain locations at a minimum cost or minimum time. Some classic situations would be firefighters responding to an alarm or an ambulance responding to a traffic accident. For such situations it is important to know ahead of time what is the fastest route between the base and where emergencies occur.

Let us consider the network given in Figure 5.6 where the number written next to each arc represents the distance[5] in metres of that arc.

[5] Or the cost or time to travel on that arc.

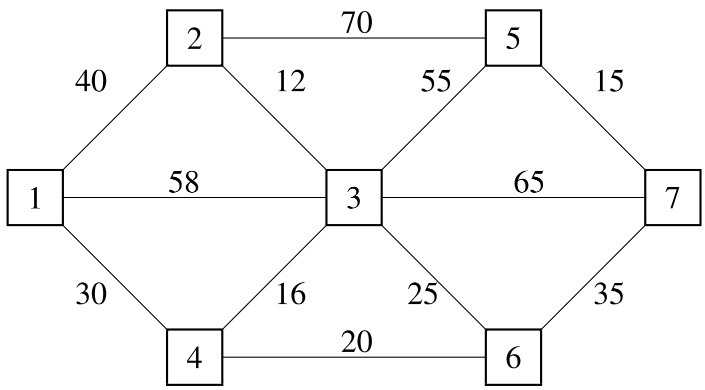


Figure 5.6: Data for the Shortest Path Example

Suppose that we want to know the shortest path from $\boxed{1}$ to $\boxed{7}$. Obviously, for such a small example it is a trivial matter to find the optimal solution, which is $\boxed{1}$ $\rightarrow \boxed{4} \rightarrow \boxed{6} \rightarrow \boxed{7}$. However, we want an efficient solution procedure which can solve problems of any size.

5.7.1 LP Model and Excel Solution

Each arc in the network is either part or not part of the shortest path, so we define:

$$X_{ij} = \left\{ \begin{array}{ll} 1 & \text{if arc } i,j \text{ is part of the shortest path} \\ 0 & \text{otherwise} \end{array} \right\} \quad \text{all defined arcs } i,j$$

The objective is to minimize the total distance travelled on the path from the beginning to the end:

$$\begin{array}{ll} \text{minimize} & 40X_{1,2} + 58X_{1,3} + 30X_{1,4} + 40X_{2,1} + 12X_{2,3} + 70X_{2,5} + 58X_{3,1} + 12X_{3,2} + \\ & 16X_{3,4} + 55X_{3,5} + 25X_{3,6} + 65X_{3,7} + 30X_{4,1} + 16X_{4,3} + 20X_{4,6} + 70X_{5,2} + \\ & 55X_{5,3} + 15X_{5,7} + 25X_{6,3} + 20X_{6,4} + 35X_{6,7} + 65X_{7,3} + 15X_{7,5} + 35X_{7,6} \end{array}$$

If we send one unit from the beginning node to the ending node (in this example, these are nodes 1 and 7 respectively) then the net flow at each node (i.e. the total flow in minus the total flow out) must be -1 at the beginning, 1 at the end, and 0 at every other node. Hence the constraints are:

$$
\begin{array}{rrcr}
\text{Beginning at Node 1} & X_{2,1}+X_{3,1}+X_{4,1}-X_{1,2}-X_{1,3}-X_{1,4} & = & -1 \\
\text{Node 2} & X_{1,2}+X_{3,2}+X_{5,2}-X_{2,1}-X_{2,3}-X_{2,5} & = & 0 \\
\text{Node 3} & X_{1,3}+X_{2,3}+X_{4,3}+X_{5,3}+X_{6,3}+X_{7,3} & & \\
 & -X_{3,1}-X_{3,2}-X_{3,4}-X_{3,5}-X_{3,6}-X_{3,7} & = & 0 \\
\text{Node 4} & X_{1,4}+X_{3,4}+X_{6,4}-X_{4,1}-X_{4,3}-X_{4,6} & = & 0 \\
\text{Node 5} & X_{2,5}+X_{3,5}+X_{7,5}-X_{5,2}-X_{5,3}-X_{5,7} & = & 0 \\
\text{Node 6} & X_{3,6}+X_{4,6}+X_{7,6}-X_{6,3}-X_{6,4}-X_{6,7} & = & 0 \\
\text{Ending at Node 7} & X_{3,7}+X_{5,7}+X_{6,7}-X_{7,3}-X_{7,5}-X_{7,6} & = & 1
\end{array}
$$

Putting this onto Excel, we use square arrays for the variables, and for the distances. For the benefit of readability only, node 1 is green in column A and node 7 is green in row 2, indicating that these are the beginning and ending nodes. On the main diagonal of the distance matrix, all the numbers are zeroes. Off the main diagonal, the actual distance is used for all arcs that are defined, such as arc 1,3 for which the distance is 40 metres. For the undefined arcs, such as 1,5, the "dummy" distance of 9999 metres is used. This high number acts as a penalty cost which will prevent the arc from being selected for the shortest path, because the distance is prohibitive. In the picture which follows, the variable cells are in the range B3:H9. This range contains 49 cells, but only the ones highlighted in yellow, which represent defined arcs, can form part of the solution. The rows are summed in column I using `=SUM(B3:H3)` in cell I3, which is copied into the range I3:I9. The columns are summed in row 10, using `=SUM(B3:B9)` in cell B10, which is copied into the range B10:H10. The challenge is to find the correct formulas for the Net Flow in column J. Here are two ways to do this:

1. The slow way is to manually enter a formula in each cell in the range J3:J9. The Net Flow is the Flow In − the Flow Out. Hence we put =B10-I3 in cell J3, =C10-I4 in cell J4, and so on, finally putting =H10-I9 in cell J9.

2. The faster way requires a knowledge of Excel's **TRANSPOSE** function. This can be used to rotate the horizontal range B10:H10 by 90 degrees clockwise. This transposed row therefore becomes a column, and now the column I3:I9 can be subtracted from it.

   However, using this function is more complicated than using most Excel functions.

   - First we need to click on cell J3 and then drag the mouse down to cell

J9; this creates a blank cell in J3 but all the other cells in this range will be in grey or black.

- Secondly, into cell J3 we enter the formula =TRANSPOSE(B10:H10)-I3:I9.

- Thirdly, we must press the ***Control*** key (and keep it held down), then the ***Shift*** key (and keep it held down), and then finally press the ***Enter*** key.

Although only the cells in yellow are variable cells, things are made easier if we use the entire range B3:H9.[6] In cell J14, the objective function is computed by using the SUMPRODUCT function to multiply the cells in B3:H9 by the corresponding cells in the range B13:H19. This is possible because of the 9999 penalty costs which will prevent the non-variable cells from being chosen.

We use the Solver to minimize cell J14, by changing variable cells B3:H9, subject to J3:J9 = L3:L9.

[6] Alternatively, we could define only the yellow cells to be the changing variable cells. However, it takes longer to do it this way.

	A	B	C	D	E	F	G	H	I	J	K	L
1		Flows Between Nodes								Net		
2	From \ To	1	2	3	4	5	6	7	Out	Flow		RHS
3	1	0	0	0	1	0	0	0	1	-1	=	-1
4	2	0	0	0	0	0	0	0	0	0	=	0
5	3	0	0	0	0	0	0	0	0	0	=	0
6	4	0	0	0	0	0	1	0	1	0	=	0
7	5	0	0	0	0	0	0	0	0	0	=	0
8	6	0	0	0	0	0	0	1	1	0	=	0
9	7	0	0	0	0	0	0	0	0	1	=	1
10	In	0	0	0	1	0	1	1				
11		Distances (in metres) Between Nodes										
12	From \ To	1	2	3	4	5	6	7		Shortest		
13	1	0	40	58	30	9999	9999	9999		Distance		
14	2	40	0	12	9999	70	9999	9999		85		
15	3	58	12	0	16	55	25	65				
16	4	30	9999	16	0	9999	20	9999				
17	5	9999	70	55	9999	0	9999	15				
18	6	9999	9999	25	20	9999	0	35				
19	7	9999	9999	65	9999	15	35	0				

We see that the shortest path has a distance of 85 metres. The shortest path itself is found by following all the variable cells which contain the number 1. The shortest path is seen to be $\boxed{1} \to \boxed{4} \to \boxed{6} \to \boxed{7}$.

Having found the shortest path between nodes 1 and 7, if we now wish to find the shortest path between a different pair of nodes, not much work needs to be done on the user's part. Suppose that we wish to know the shortest path between nodes 4 and 5. There is no change to the objective function. In the constraints, node 4 rather than node 1 has a -1 on the right-hand side, and node 5 rather than node 7 has a 1 on the right-hand side.

$$\begin{array}{rrcr}
\text{Node 1} & X_{2,1}+X_{3,1}+X_{4,1}-X_{1,2}-X_{1,3}-X_{1,4} & = & 0 \\
\text{Node 2} & X_{1,2}+X_{3,2}+X_{5,2}-X_{2,1}-X_{2,3}-X_{2,5} & = & 0 \\
\text{Node 3} & X_{1,3}+X_{2,3}+X_{4,3}+X_{5,3}+X_{6,3}+X_{7,3} & & \\
 & -X_{3,1}-X_{3,2}-X_{3,4}-X_{3,5}-X_{3,6}-X_{3,7} & = & 0 \\
\text{Beginning at Node 4} & X_{1,4}+X_{3,4}+X_{6,4}-X_{4,1}-X_{4,3}-X_{4,6} & = & -1 \\
\text{Ending at Node 5} & X_{2,5}+X_{3,5}+X_{7,5}-X_{5,2}-X_{5,3}-X_{5,7} & = & 1 \\
\text{Node 6} & X_{3,6}+X_{4,6}+X_{7,6}-X_{6,3}-X_{6,4}-X_{6,7} & = & 0 \\
\text{Node 7} & X_{3,7}+X_{5,7}+X_{6,7}-X_{7,3}-X_{7,5}-X_{7,6} & = & 0
\end{array}$$

On the Excel file, node 4 is green in column A and node 5 is green in row 2, indicating that these are the new beginning and ending nodes. Of course, the colouring is just a label for the user; Excel understands that the model has changed by altering column L. The -1 now goes in cell L6, and the 1 is placed in cell L7. Re-solving the model, we obtain:

	A	B	C	D	E	F	G	H	I	J	K	L
1		Flows Between Nodes								Net		
2	From \ To	1	2	3	4	5	6	7	Out	Flow		RHS
3	1	0	0	0	0	0	0	0	0	0	=	0
4	2	0	0	0	0	0	0	0	0	0	=	0
5	3	0	0	0	0	0	0	0	0	0	=	0
6	4	0	0	0	0	0	1	0	1	-1	=	-1
7	5	0	0	0	0	0	0	0	0	1	=	1
8	6	0	0	0	0	0	0	1	1	0	=	0
9	7	0	0	0	0	1	0	0	1	0	=	0
10	In	0	0	0	0	1	1	1				
11		Distances (in metres) Between Nodes										
12	From \ To	1	2	3	4	5	6	7		Shortest		
13	1	0	40	58	30	9999	9999	9999		Distance		
14	2	40	0	12	9999	70	9999	9999		70		
15	3	58	12	0	16	55	25	65				
16	4	30	9999	16	0	9999	20	9999				
17	5	9999	70	55	9999	0	9999	15				
18	6	9999	9999	25	20	9999	0	35				
19	7	9999	9999	65	9999	15	35	0				

The distance along the shortest path is 70 metres, and the path is $\boxed{4} \to \boxed{6} \to \boxed{7} \to \boxed{5}$.

5.8 Summary

This chapter presented several types of network problems, though the first three, the assignment, transportation, and transshipment problems, were presented without reference to the underlying network structure. In the assignment problem we seek the minimum cost of assigning n items of one type to n items of another. In the transportation problem, we seek to minimize the cost of sending units from supply points to points of demand. The transshipment problem is a variant of the transportation problem, in which some points may be both origins and destinations. Though specialized algorithms exist for these three problems, all were solved here as linear programming models using the Excel Solver.

We then introduced the concept of a network, which consists of nodes and arcs, and presented three network problems – the minimum spanning tree problem, the maximal flow problem, and the shortest path problem. The minimum spanning problem is that of connecting (directly or indirectly) each node with each other node at minimum cost. We presented a simple visual algorithm for this problem. In the maximum flow problem, some or all of the arcs have capacity constraints. Given these constraints, we wish to know the upper limit to the quantity which can be shipped between a given pair of nodes. We formulated this problem algebraically, and solved it using Excel. In the shortest path problem the physical network of nodes and arcs is already in place, and we seek the shortest (distance, cost, or time) path from one given node to another. This was modelled using linear programming and then solved using the Excel Solver.

5.9 Problems for Student Completion

5.9.1 Assignment Problem

Chess matches are most interesting when the two players are approximately equal in ability. There are eight players from two teams, whose scores based on past performance are: Team 1 – 1600, 1825, 1670, and 1710; Team 2 – 1920, 1750, 1660, and 1790. For the first round, the tournament organizers want to see close match-ups.

(a) Formulate an assignment model for deciding the players for the four matches.

(b) Solve the problem using the Excel Solver.

5.9.2 Transportation/Transshipment Problem

A company makes smart telephones at facilities in Waterloo (Canada), Cambridge (England), and Mumbai (India). These plants can make 1200, 900, and 2400 telephones per week beyond the demand in the "local" markets of Canada/USA, Europe, and Western Asia respectively. All three plants can ship to markets elsewhere: Latin America, Africa, and Eastern Asia. The demands per week in these three markets are for 500, 1400, and 2500 telephones per week respectively. Phones are shipped in boxes of 100. The shipping costs per box are as follows:

From/To	Latin America	Africa	Eastern Asia
Waterloo	200	340	270
Cambridge	290	250	310
Mumbai	300	240	250

(a) Formulate a model and solve using Excel to determine how much should be shipped from the factories to the markets.

(b) Now suppose that phones can be shipped from Waterloo to Cambridge at a cost of $40 per box. Formulate and solve the new model.

5.9.3 Minimum Spanning Tree

A cable TV company needs to run some wires to serve six customers who are located a considerable distance apart. The following symmetric table gives the cost (in tens of dollars) of running a direct cable between customers (an impossible or prohibitively costly connection is indicated as –):

From/To	1	2	3	4	5	6
1	–	39	41	62	40	–
2	39	–	50	38	60	65
3	41	50	–	35	36	61
4	62	38	35	–	32	48
5	40	60	36	32	–	46
6	–	65	61	48	46	–

(a) Draw a picture of the six customers, showing each potential connection with its cost written next to the link.

(b) On this picture, beginning with customer 1, use the visual algorithm to find the minimum cost solution.

5.9.4 Maximal Flow Problem

The following table gives the maximum flow between nodes which are physically connected:

From/To	1	2	3	4	5	6
1	–	12	14	–	16	–
2	–	–	4	15	12	–
3	12	11	–	14	19	4
4	–	21	–	–	16	9
5	–	–	11	14	–	22
6	–	–	–	13	18	–

We wish to determine the maximum flow through the network from node 2 to node 5.

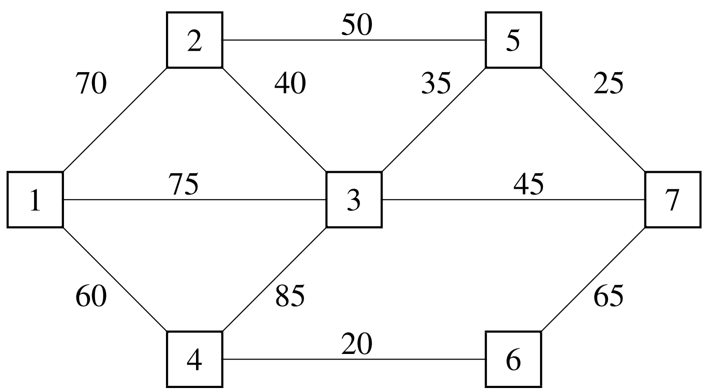
(a) Draw a picture of this situation, including the dummy arc.

(b) Formulate this problem as a linear optimization model.

(c) Solve the problem in Excel.

5.9.5 Shortest Path Problem

The following picture gives highway distances in kilometres between a set of cities. A hurricane has washed out the road that used to exist between cities 3 and 6, and so it is not shown on the map.



(a) Formulate an algebraic model for determining the shortest path between cities 2 and 6.

(b) Use the Excel Solver to determine the solution.

Chapter 6

Integer Models

In Chapter 3 we noted that some of the variables in some of the models had to be integer. Such variables were indicated either by stating that they must be integer (in addition to being ≥ 0), or by stating that they must be $\in \{0, 1, 2, \ldots\}$. Then, to solve such models, we declared them to be *int* in the Excel Solver. We explore the issue of integer variables further in this chapter, and in particular we examine:

1. The graphical solution of two-variable integer models.

2. The formulation of models in which some of the variables must be either 0 or 1, i.e. $\in \{0, 1\}$, and the *bin* declaration in the Solver for solving such models.

6.1 Introduction

6.1.1 Removing the Assumption of Real-Numbered Variables

One of the four assumptions of linear optimization models is that the each variable must be allowed to be a real number (e.g. 5.0, 6.11111... or 8.3) rather than be required to be an integer. However, there are many situations where this assumption is not valid. In such cases, unless the linear solution is naturally integer, the user is unsure how the decision variables should be treated. For example, if a linear model recommends that a firm purchase 7.92 trucks, should this be rounded to 8 (the nearest integer), rounded down to 7, or some other solution? Indeed, there may be no feasible solution when the restriction of integrality is added to the model. In addition to this type of situation, managers often wish to model either/or type decisions, which are typically represented by a variable which must be either 0 or 1.

In this chapter we examine what happens when the assumption of real-numbered variables is removed. With this assumption gone, we are in a situation where some or all of the variables of a model are restricted to the set of integers.[1] If *some*, but not all, the variables are required to be integer then we are dealing with a *mixed* integer model. If *all* the variables are required to be integer then we have a *pure* integer model.

As far as the *formulation* is concerned, the easiest situation to handle is where a variable, which arises naturally out of a formulation, must be integer rather than continuous. In such a situation, the non-negativity restriction is merely replaced by a restriction that this variable must be in the set of positive integers.

For example, suppose that H_7 represents the number of workers to be hired in month 7. The formulation proceeds as in the linear case, except that at the end, instead of writing $H_7 \geq 0$, we write[2]

$$H_7 \in \{0, 1, 2, 3, \ldots\}$$

If the number of workers to be hired is restricted to say 20, then $H_7 \leq 20$ is a constraint; we do not need to end the set of integers at 20.

6.1.2 Naturally Integer Solutions and Rounding

There are some integer models which, when solved as if they were linear models, give a solution which obeys the integrality restrictions. Such models are said to be *naturally* integer. Important cases of this are the *assignment* and *transportation* problems. Problems which have left hand side coefficients of -1, 0, or 1, and right hand side coefficients which are integers, are often naturally integer, but these conditions are neither necessary nor sufficient. In general, models with arbitrary structure are highly unlikely to be naturally integer.

For problems which are not naturally integer, we must proceed further. For some problems, it may not be necessary for all practical purposes to try to find the optimal solution. In a model in which the variables have a high numerical value, for example $X_1 = 732.91$, the optimal integer value for this variable might well

[1]Or a finite subset of the set of integers. There is no loss in generality in excluding the possibility of discrete fractional values, since by a transposition of variables we can always create integer-valued variables.

[2]This is read as "H_7 is in the set of numbers 0, 1, 2, and so on."

be $X_1^* = 732$ or 733. Even if neither of these is optimal, if one of the solutions is feasible then it may be nearly optimal. By "nearly", we mean that the OFV is near its optimal value. Any solution so obtained should of course be checked for feasibility (i.e. we must verify that it satisfies the constraints.)

Rounding a linear solution to obtain an integer solution is an example of a *heuristic*. A heuristic is an approach for solving a problem which hopefully gives a good solution but does not necessarily give an optimal solution. When using a heuristic it is desirable to know a bound for the gap between it and the optimal solution. For example, suppose that we solve a maximization problem by ignoring the integrality restrictions and we find that the optimal OFV (linear) is \$746,831.29. Suppose now that we round this solution to obtain integer values and we find a feasible solution whose OFV is \$746,688.10. For the *optimal* solution to the integer model it follows that

$$\$746,688.10 \leq \text{OFV}^* \leq \$746,831.29$$

Hence the heuristic is no worse than \$143.19 below the optimum. It is knowing this sort of information that can give a decision maker confidence in the recommendation, even though he or she is aware that the solution is not guaranteed to be optimal.

Certainly, the heuristic of rounding should be avoided in any of the following three situations:

- the rounded solution is not feasible
- the percentage of change involved with rounding is large (e.g. rounding 2.1 to 2 is almost a 5% drop)
- any time the exact optimal solution must be obtained.

In any of these situations, an algorithm which handles the integrality restrictions is required. One such general approach is the branch-and-bound algorithm which is described in Appendix C beginning on page 492. Briefly, the branch and bound algorithm solves a set of sub-problems, each of which is a linear model solvable by the simplex algorithm. Whenever some of the variables are required to be integer, the sensitivity analysis presented earlier is not valid. This algorithm is built-in to the Solver in Excel, and it also available in many dedicated software packages for linear and integer programming.

6.1.3 Solution by Computer

A spreadsheet solver can be used to solve integer optimization models. There are separate declarations for general integer variables ($\in \{0,1,2,3,\ldots\}$), and 0/1 integer variables ($\in \{0,1\}$).

As first explained in the telephone operator problem of Chapter 3, the procedure in the Excel Solver for declaring variables to be general integer variables, i.e. they are $\in \{\ldots, \ -3,-2,-1,0,1,2, \ \ldots\}$ is given below. If the "Make Unconstrained Variables Non-Negative" box is ticked, then the joint effect is to declare the variables to be positive integers, i.e. they are $\in \{0,1,2,3, \ \ldots\}$.

1. Open the Solver, and click on "Add".
2. The "Add Constraint" dialog box appears, with a blinker in the space below "Cell Reference:".
3. Use the mouse to highlight the variable cells. The range with dollar signs will appear in the space.
4. In the middle where the "<=" appears, click on the down arrow to the right, and then click on "int".
5. The "<=" will be replaced by "int", and "integer" will appear in the space to the right.
6. Click on "OK".
7. In the "Solver Parameters" dialog box, *range* `= integer` will appear in the "Subject to the Constraints" section (where the cell references for the variable cells are displayed for *range*).
8. Click on the Solve button.

Also, a cell (or range of cells) can be declared as binary, meaning that the only possible values are 0 or 1. Similar to the above, such declarations are made under Solver in the Add constraint dialog box, by selecting *bin* in the middle section.

6.2 Models with Two General Integer Variables

As with the linear case, models with just two integer variables can be solved graphically. Some examples follow.

6.2.1 Loading Boxes onto a Cargo Plane

Here we consider the example of loading boxes onto a cargo plane which we saw at the end of Chapter 1.

Description

> Two types of big boxes are about to be loaded onto a small cargo plane. A Type 1 box has a volume of 2.9 cubic metres (m^3), and a mass of 470 kilograms (kg), while a Type 2 box has a volume of 1.8 m^3 and a mass of 530 kg. There are six Type 1 boxes and eight Type 2 boxes waiting to be loaded. There is only one cargo plane, and it has a volume capacity of 15 m^3 and a mass capacity of 3600 kg. Obviously, not all the boxes can be put onto the plane, therefore suppose that the objective is to maximize the value of the load. We will consider the following three situations: (i) both type of boxes are worth \$400 each; (ii) a Type 1 box is worth \$600, and a Type 2 box is worth \$250; and (iii) a Type 1 box is worth \$300, and a Type 2 box is worth \$750.

Back then, we used an enumerative method to find all potential solutions, and then evaluated each of these to find the optimal ones. Now, we will formulate and solve this problem using integer optimization.

Formulation

We need to determine how many boxes of each type are carried on the plane, so we define:

$$
\begin{aligned}
X_1 &= \text{the number of Type 1 boxes carried on the plane} \\
X_2 &= \text{the number of Type 2 boxes carried on the plane}
\end{aligned}
$$

There are three cases of profit data. Each gives rise to a different objective function:

$$
\begin{aligned}
\text{(i)} \quad & \text{maximize} \quad 400X_1 + 400X_2 \\
\text{(ii)} \quad & \text{maximize} \quad 600X_1 + 250X_2 \\
\text{(iii)} \quad & \text{maximize} \quad 300X_1 + 750X_2
\end{aligned}
$$

There is a constraint for the volume capacity of the plane. By now it should be easy to write this constraint:

$$\text{Volume} \;\; 2.9X_1 + 1.8X_2 \leq 15$$

Next, there is a constraint for the mass capacity of the plane:

$$\text{Mass} \;\; 470X_1 + 530X_2 \leq 3600$$

The plane cannot carry more boxes than are available to be carried, therefore we have two more constraints:

$$\text{Type 1} \; X_1 \leq 6$$

and

$$\text{Type 2} \; X_2 \leq 8$$

In this example, we have not only the non-negativity restrictions, but also the requirement that both variables must be integer. The complete formulation is therefore:

$$\begin{aligned} X_1 &= \text{the number of Type 1 boxes carried on the plane} \\ X_2 &= \text{the number of Type 2 boxes carried on the plane} \end{aligned}$$

$$\begin{array}{rrcrcr}
\text{One of:} & & & & & \\
\text{(i) maximize} & 400X_1 & + & 400X_2 & & \\
\text{(ii) maximize} & 600X_1 & + & 250X_2 & & \\
\text{(iii) maximize} & 300X_1 & + & 750X_2 & & \\
\text{subject to} & & & & & \\
\text{Volume} & 2.9X_1 & + & 1.8X_2 & \leq & 15 \\
\text{Mass} & 470X_1 & + & 530X_2 & \leq & 3600 \\
\text{Type 1} & X_1 & & & \leq & 6 \\
\text{Type 2} & & & X_2 & \leq & 8 \\
\text{non-negativity} & X_1 & , & X_2 & \geq & 0 \\
\text{integer} & X_1 & , & X_2 & &
\end{array}$$

We begin as always by making a grid and plotting the boundaries of the constraints. Although the last two constraints tell us that the optimal solution must be contained within a 6 by 8 grid, we will see that a slightly larger 8 by 9 grid allows us to show all of the volume and mass constraints.

The boundary of the volume constraint is:

$$2.9X_1 + 1.8X_2 = 15$$

Setting $X_1 = 0$ makes $1.8X_2 = 15$, and hence $X_2 \approx 8.333$. Setting $X_2 = 0$ makes $2.9X_1 = 15$, and hence $X_1 \approx 5.172$.

The boundary of the mass constraint is:

$$470X_1 + 530X_2 = 3600$$

Setting $X_1 = 0$ makes $530X_2 = 3600$, and hence $X_2 \approx 6.792$. Setting $X_2 = 0$ makes $470X_1 = 3600$, and hence $X_1 \approx 7.660$.

The Type 1 constraint's boundary is a vertical line through 6, and the boundary of the Type 2 constraint is a horizontal line through 8.

In summary we have:

Constraint	First Point	Second Point
Volume	(0,8.333)	(5.172,0)
Mass	(0,6.792)	(7.660,0)
Type 1	$X_1 = 6$	vertical
Type 2	$X_2 = 8$	horizontal

All the arrows are easy; the origin is true for every constraint, so every arrow points toward the origin.

These four constraints, along with their arrows and word descriptions, are shown in Figure 6.1.

We can now fill-in with colour the region in which all four constraints and the two non-negativity restrictions are true. This region is shown in gold in Figure 6.2.

Because the variables must be integer, only those points in the coloured area which represent integer values for both variables are feasible.[3] Finding all these points, which we represent as dots, is fairly easy except when a point is very near one of the constraint boundaries. In this example, the points (1,6), (2,5) are near the boundary of the mass constraint, and the point (4,2) is near the boundary of the volume constraint. We can test these contentious points by substituting the values into the appropriate constraint. For example, for the point (1,6):

$$\begin{aligned} 470(1) + 530(6) &= 470 + 3180 \\ &= 3650 \\ &\not\leq 3600 \end{aligned}$$

[3]If X_1 were integer, but X_2 not integer, then we would have a set of feasible vertical lines. If X_2 were integer, but X_1 not integer, then we would have a set of feasible horizontal lines.

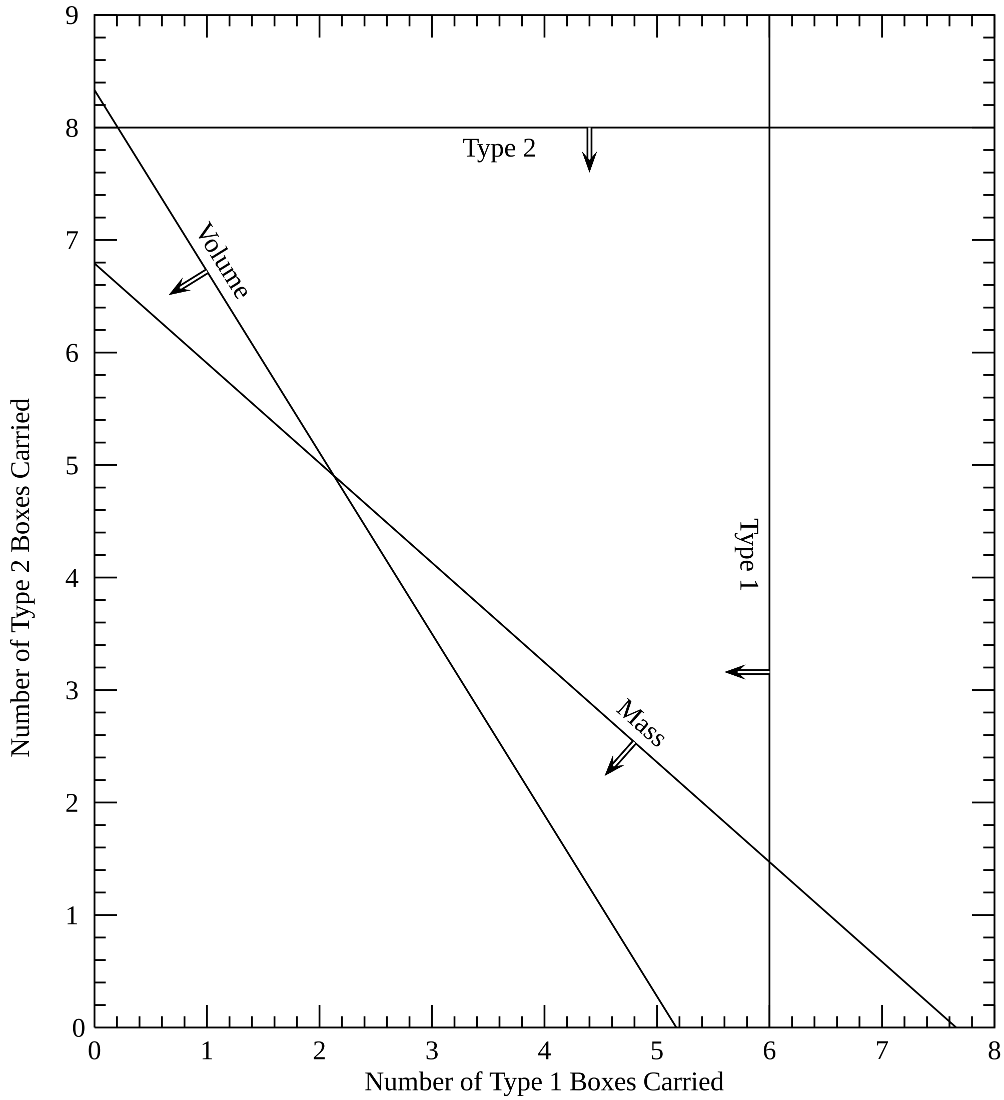


Figure 6.1: Cargo Plane Problem – Constraints

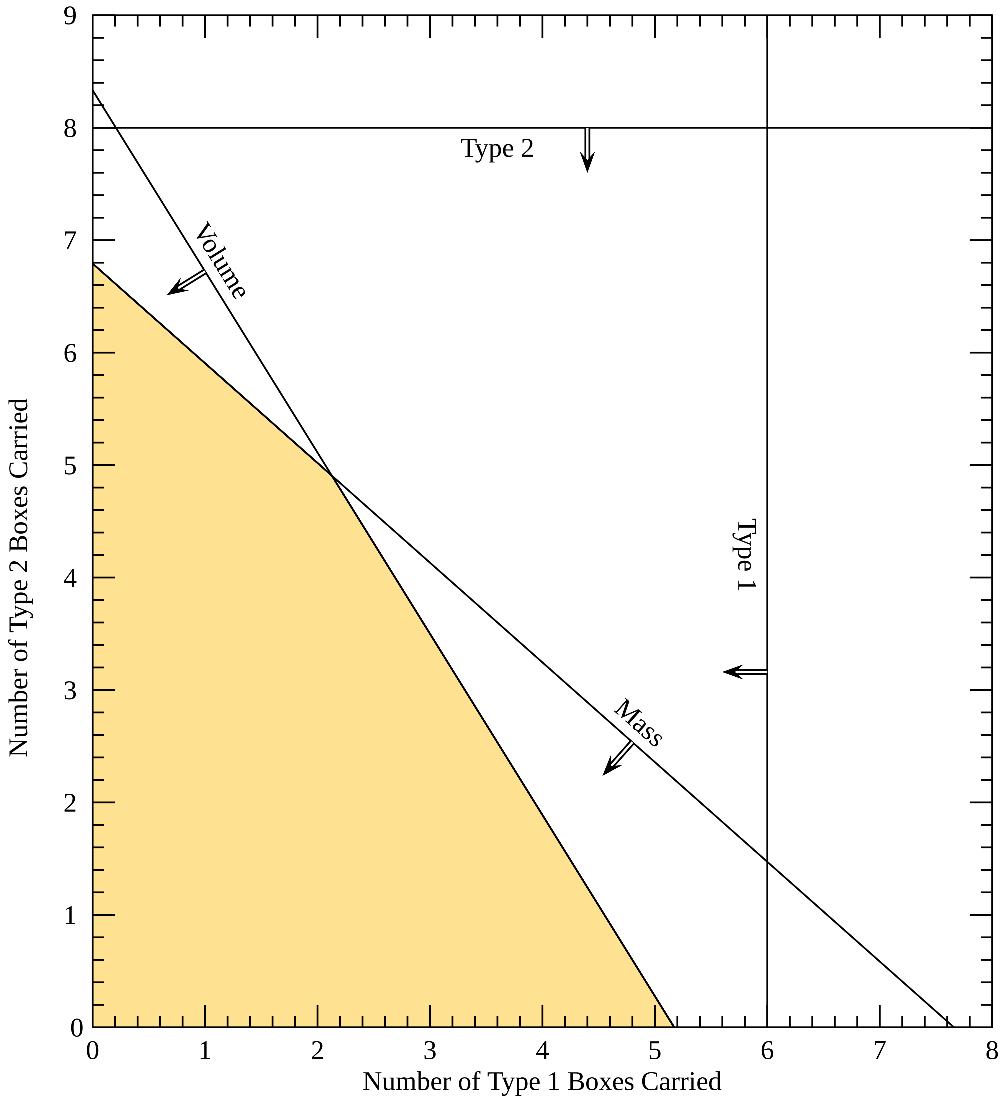


Figure 6.2: Cargo Plane Problem – Non-Integer Region

Hence the point (1,6) is infeasible, and is therefore excluded from consideration. On the other hand, for the point (2,5) we obtain:

$$\begin{aligned} 470(2)+530(5) &= 940+2650 \\ &= 3590 \\ &\leq 3600 \ \surd \end{aligned}$$

Therefore, the point (2,5) is feasible. Finally, for the point (4,2) we use the volume constraint:

$$\begin{aligned} 2.9(4)+1.8(2) &= 11.6+3.6 \\ &= 15.2 \\ &\not\leq 15 \end{aligned}$$

We see that the point (4,2) is infeasible, and it is therefore excluded. We are left with 26 feasible points, which are shown in Figure 6.3.

Beginning with the first of the three objective functions, we seek to maximize $400X_1 + 400X_2$. The shortcut produces points which are off the graph, but dividing by 100 gives the points 4 on the vertical axis and 4 on the horizontal axis. These are connected to form the first of three trial lines. We then move the rolling ruler, stopping not at the corner of the volume and mass constraints (because this point is infeasible), but instead at the integer solution (2,5). This is shown in Figure 6.4.

The optimal objective function value is:

$$\$400(2)+\$400(5) = \$2800$$

If an integer solution had not been required, we would have obtained a solution at the corner of the volume and mass constraints. By using linear algebra we would have found $X_1^* \approx 2.12735$, $X_2^* \approx 4.90593$, and $\text{OFV}^* \approx \$2813.31$. Since we do require integer values we have instead $X_1^* = 2$, $X_2^* = 5$, and $\text{OFV}^* = \$2800.00$. By imposing the requirement that the variables be integer, we have impaired the objective function value by \$13.31. This will always be true – for models which are not naturally integer, adding a requirement that the variables must be integers will impair (i.e. lower for a maximization model, higher for a minimization model) the objective function value.

Using the same diagram we draw the trial and optimal isovalue lines for situations (ii) (in green) and (iii) (in blue). As before, in order to obtain the intercepts on the axes for the two trial lines, the objective function coefficients were divided

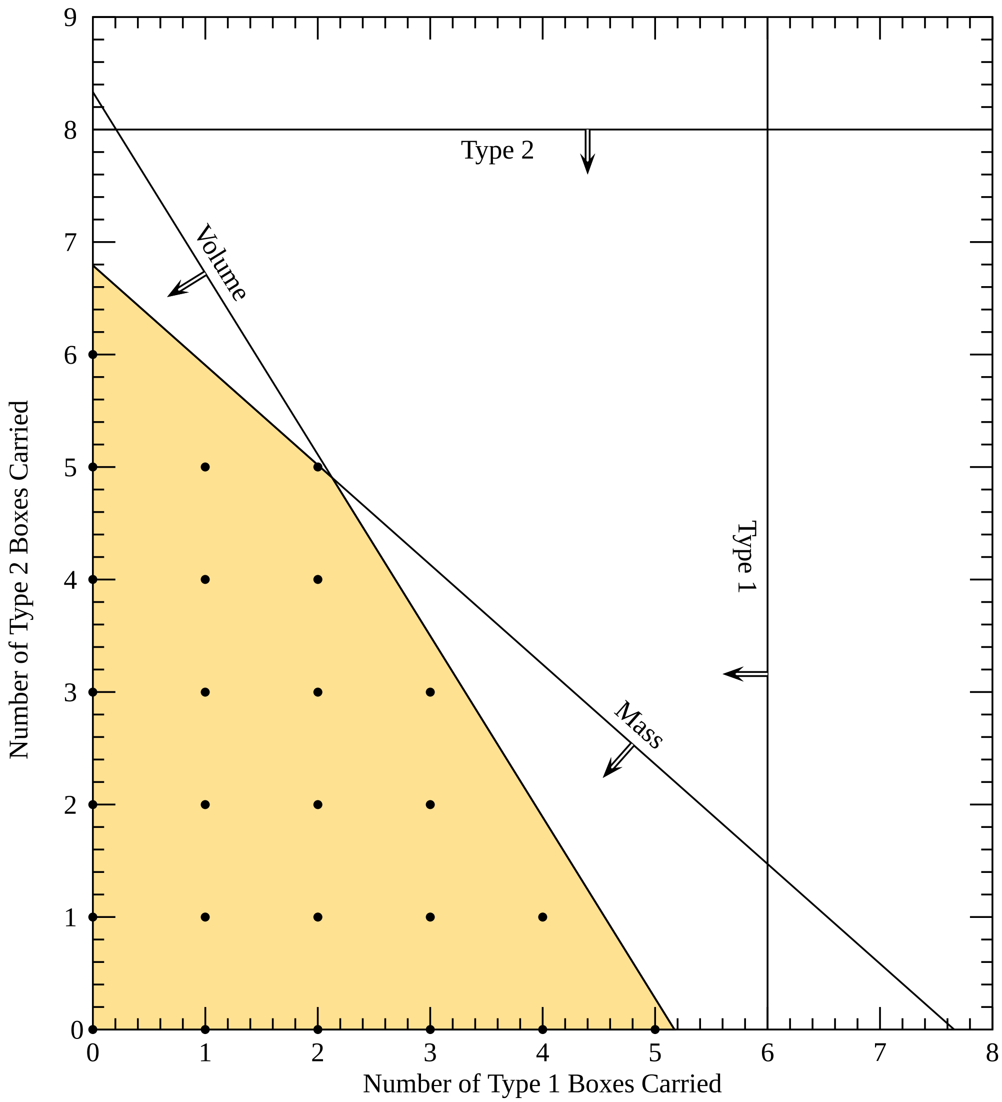


Figure 6.3: Cargo Plane Problem – Set of Feasible Points

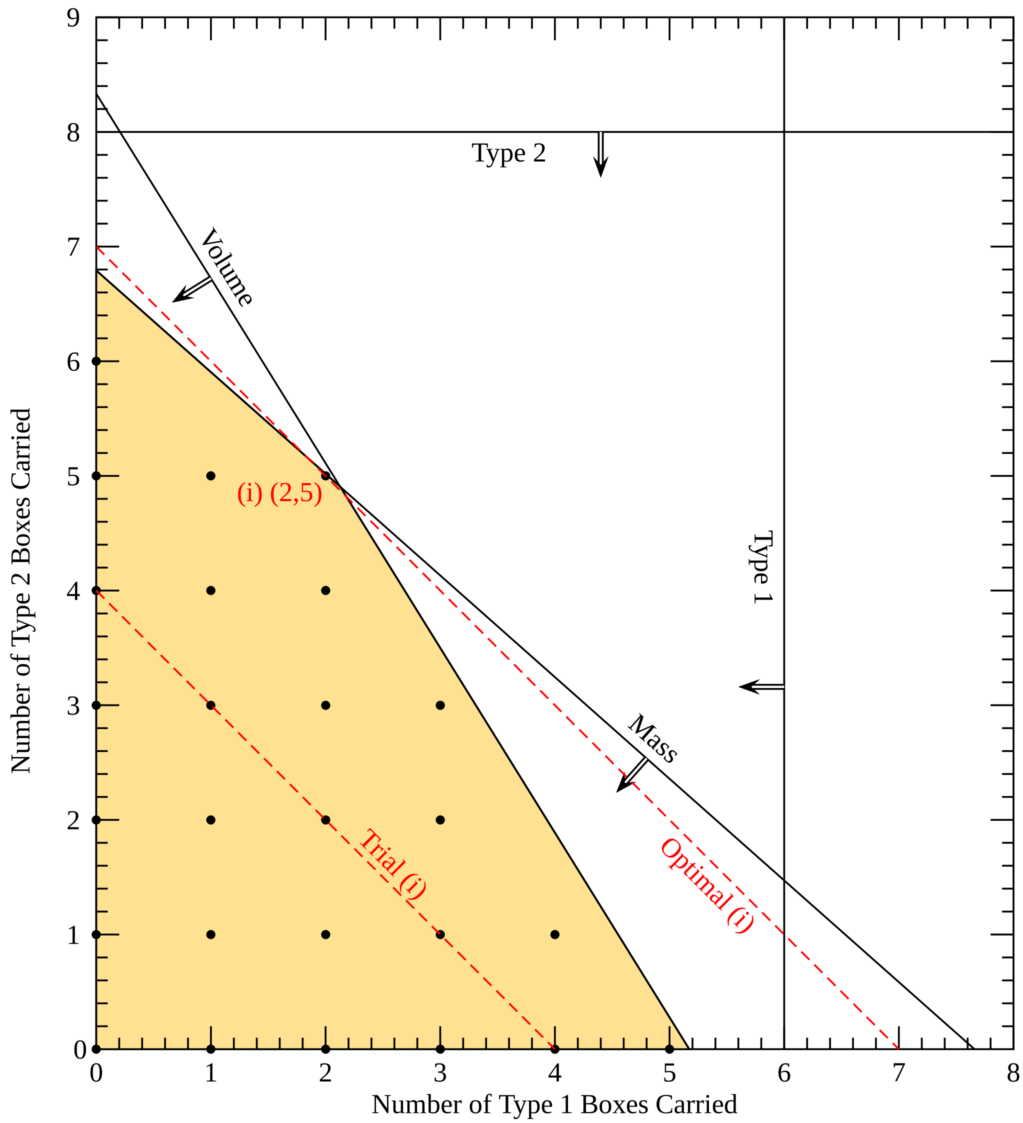


Figure 6.4: Cargo Plane Problem – Optimal Solution for Part (i)

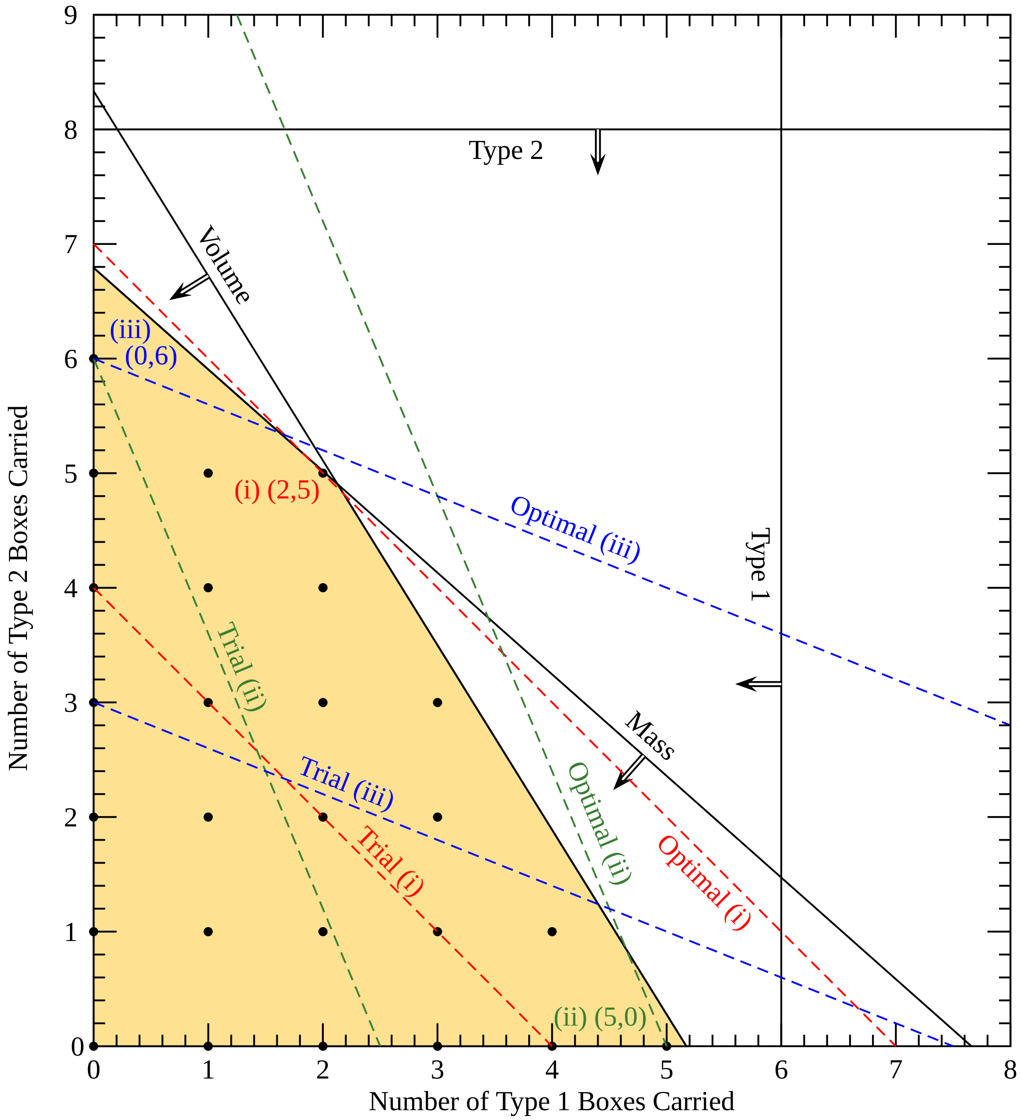


Figure 6.5: Cargo Plane Problem – Optimal Solution for Parts (i), (ii), (iii)

by 100. All this is shown in Figure 6.5. We identify the optimal solution for case (ii) as (5,0), i.e. five boxes of Type 1 only, the OFV is $\$600(5)+\$250(0)=\$3000$. The optimal solution for case (iii) is (0,6), i.e. six boxes of Type 2 only, the OFV is $\$300(0)+\$750(6)=\$4500$. In summary we have the recommendation given in the following table:

Situation	Profit per Box		Optimal Load		Total
	Type 1	Type 2	Type 1	Type 2	Profit
(i)	400	400	2	5	\$2800
(ii)	600	250	5	0	\$3000
(iii)	300	750	0	6	\$4500

6.2.2 A Pure Integer Example with Negative LHS Coefficients

Algebraic Model The pure integer model below repeats some of what was discussed in the Cargo Plane example, but the constraints are more challenging to graph, because some of the left-hand side coefficients are negative.

$$
\begin{array}{rrcrcl}
\text{maximize} & 4X_1 & + & 3X_2 & & \\
\text{subject to} & & & & & \\
(1) & -X_1 & + & 6X_2 & \leq & 18 \\
(2) & -2X_1 & + & 5X_2 & \geq & 10 \\
& & & & & \\
& X_1 & , & X_2 & \in & \{0,1,2,3,\ldots\}
\end{array}
$$

Points for the Boundaries of the Constraints We can proceed as we would for a linear problem, drawing the axes, the boundaries of the two constraints, and so on. However, finding the points for the boundary lines is complicated by the presence of negative left-hand side coefficients.

1. To plot the boundary of (1), we set $-X_1+6X_2=18$. If $X_1=0$, then $X_2=3$. However, when we set $X_2=0$, we obtain $-X_1=18$, and hence $X_1=-18$. Since this point is not positive, it does not help us. Instead, to find another point on the line, we add X_1 to both sides to obtain:

$$
\begin{aligned}
6X_2 &= 18+X_1 \\
X_2 &= 3+\left(\frac{1}{6}\right)X_1
\end{aligned}
$$

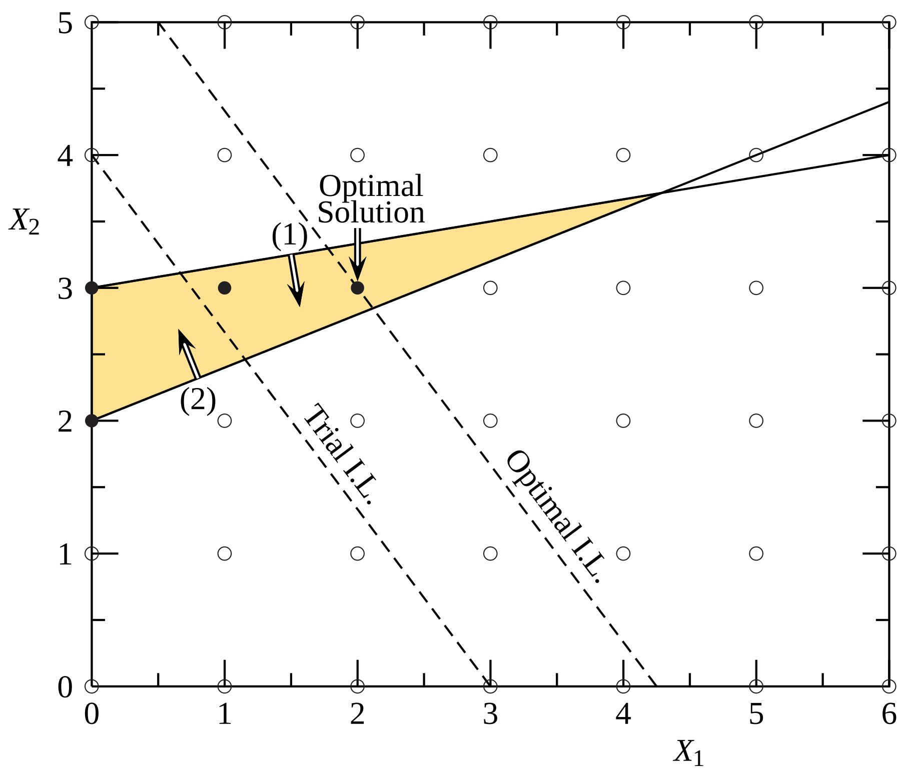


Figure 6.6: Graphical Solution of a Pure Integer Model

If we let $X_1 = 6$, we obtain $X_2 = 3+1 = 4$. Hence the boundary of (1) passes through $(X_1, X_2) = (0,3)$ and $(6,4)$. Since $-0+6(0) = 0 \leq 18$, the origin is feasible, and so the arrow points south-east.

2. To plot the boundary of (2), we set $-2X_1 + 5X_2 = 10$. If $X_1 = 0$, then $X_2 = 2$. To find another point on the line we add $2X_1$ to both sides to obtain:

$$\begin{aligned} 5X_2 &= 10 + 2X_1 \\ X_2 &= 2 + \left(\frac{2}{5}\right) X_1 \end{aligned}$$

If we let $X_1 = 5$, we obtain $X_2 = 2+2 = 4$. Hence the boundary of (2) passes through $(X_1, X_2) = (0,2)$ and $(5,4)$. Since $-2(0)+5(0) = 0 \ngeq 10$, the origin is infeasible, and so the arrow points north-west.

Completing the Solution The boundary lines of the two constraints are plotted, and we place the arrows on them. We then find the region in which both constraints and the two non-negativity restrictions are satisfied, and shade this region in gold.

As we saw with the earlier Cargo Plane example, each point where both X_1 and X_2 are integers is represented as a dot, more formally called a *lattice point*. There is a set of feasible points, which is the set of lattice points which satisfy the constraints, i.e. they are either inside or on the boundary of the gold-shaded region. In the graph shown in Figure 6.6 the lattice points are denoted by small circles; the solid ones are feasible, the open ones are infeasible.[4] This example has four feasible solutions.

A trial isovalue line is drawn between $(0,4)$ and $(3,0)$. Moving the isovalue line we see that the optimal solution to the integer model occurs at $X_1^* = 2$, $X_2^* = 3$, from which it follows that $\text{OFV}^* = 4(2)+3(3) = 17$.

Rounding Does Not Work In this example, to try to solve the model by treating it as if it were a linear example (i.e. with both variables being continuous), and then rounding each variable up or down to the next integer, would not even produce a feasible solution, much less an optimal one. The linear optimal solution, located at the boundaries of constraints (1) and (2), is $X_1 = 4\frac{2}{7}$, $X_2 = 3\frac{5}{7}$. The four possible rounded solutions are $(X_1, X_2) = (4,3)$, (4,4), (5,4), and (5,3), all of which are infeasible.

[4]There is no need to draw the infeasible lattice points; they are shown only for illustrative purposes.

6.2.3 Mixed Integer Variations

It is interesting to see what happens if one variable must be integer but the other is allowed to be continuous, thereby creating a mixed integer model.

X_1 integer, but X_2 continuous Suppose we consider the current model with $X_1 \in \{0,1,2,\ldots\}$ as before, but $X_2 \geq 0$. There is now neither a feasible region as there is when both variables are continuous, nor is there a set of lattice points as there is when both variables are integer, but instead there is a set of feasible lines which are bounded by the gold-shaded region. In this example, there are five vertical lines, one of which forms part of the vertical axis. The solution is shown in Figure 6.7.

We see visually that the optimal value of X_1 is 4, and that the vertical line intercepts the boundary of constraint (1), which is $-X_1 + 6X_2 = 18$. Substituting, we have $-4 + 6X_2 = 18$, and hence $X_2 = 3\frac{2}{3}$. At $(X_1, X_2) = (4, 3\frac{2}{3})$ the OFV is $4(4) + 3(3\frac{2}{3}) = 16 + 11 = 27$.

X_2 integer, but X_1 continuous We can easily find the solution for the situation where $X_1 \geq 0$, and $X_2 \in \{0,1,2,3,\ldots\}$ without drawing a separate picture. The only horizontal line which could be drawn is the one through $X_2 = 3$. The optimal isovalue line will pass through this horizontal line and the boundary of constraint (2). Hence we substitute $X_2 = 3$ into $-2X_1 + 5X_2 = 10$ to obtain $X_1 = 2.5$. The OFV is $4(2.5) + 3(3) = 19$.

6.3 Introduction to 0/1 Variables for Binary Choice

In this section we show, through the use of six examples (plus an optional seventh example), how to model a problem when some or all of the variables must be 0/1 integer. Examples 1 and 2 illustrate the concept of binary choice, by which many managerial decisions are modelled. Example 3 translates requirements using the words "if" or "only if" into mathematical constraints. Example 4 looks at binary choice in the context of a capacity restriction. These ideas in the context of a *fixed cost* are illustrated by Example 5. Example 6 considers these ideas where no capacity restriction is explicitly given. Optional Example 7 is a fair bit harder; it models a situation where there is a pair of constraints such that either, but not necessarily both, must be satisfied.

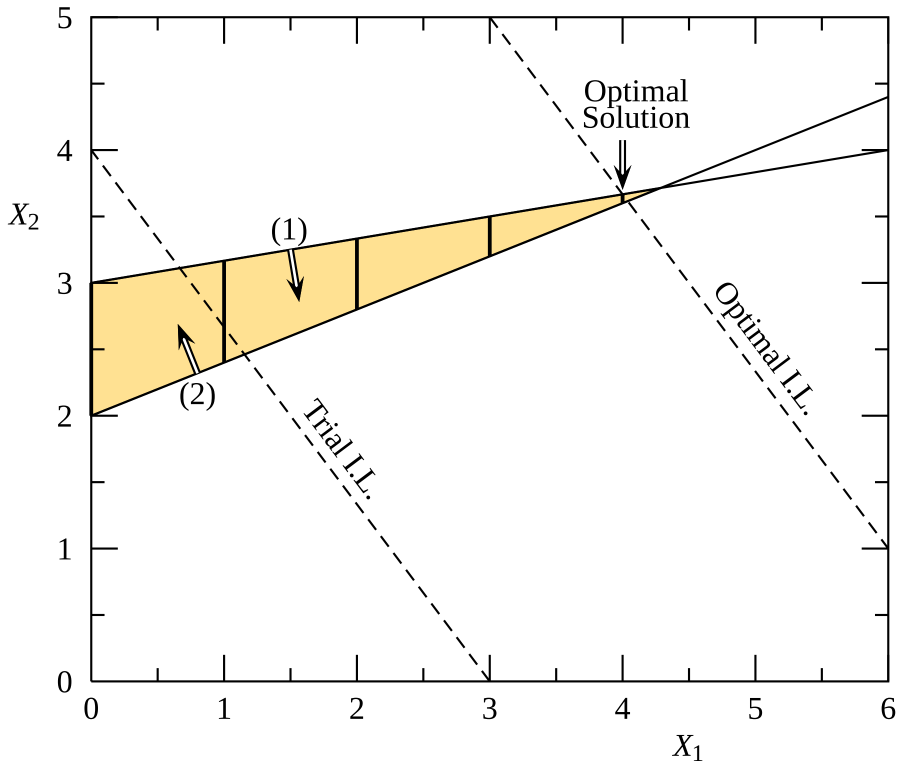


Figure 6.7: Graphical Solution of a Mixed Integer Model

6.3.1 Example 1

Suppose that a company is considering opening at least one but no more than three stores at five locations within a city. For each location we define a 0/1 variable:

$$Y_i = \left\{ \begin{array}{ll} 1 & \text{if a store is opened at location } i \\ 0 & \text{otherwise} \end{array} \right\} \qquad i = 1,2,3,4,5$$

Note the use of the word *otherwise*. This gives the negation of the previous statement, saving us from having to write "if a store is *not* opened at location i." Not all variable definitions can be shortened in this way however, as we shall see in Example 6.

We require two constraints:

$$\begin{array}{lrcl} \text{at least one store} & Y_1+Y_2+Y_3+Y_4+Y_5 & \geq & 1 \\ \text{at most three stores} & Y_1+Y_2+Y_3+Y_4+Y_5 & \leq & 3 \end{array}$$

Alternatively, we can use summation notation:

$$\begin{array}{lrcl} \text{at least one store} & \sum_{i=1}^{5} Y_i & \geq & 1 \\ \text{at most three stores} & \sum_{i=1}^{5} Y_i & \leq & 3 \end{array}$$

Instead of the non-negativity restrictions, each variable is specified to be 0/1:

$$Y_i \in \{0,1\} \qquad i = 1,2,3,4,5$$

6.3.2 Example 2

There are four items (one unit of each) awaiting shipment. We wish to transport one or more of these items in an airplane so as to maximize the total payload value, subject to a 12.3 Tonne restriction. This type of problem is called a *knapsack* problem.

Item	Weight (T)	Value
1	4.3	29.5
2	1.9	11.3
3	5.8	34.0
4	3.6	19.7

We define[5] four variables each of which can take on the value 0 or 1.

$$Y_i = \left\{ \begin{array}{ll} 1 & \text{if item } i \text{ is carried on the airplane} \\ 0 & \text{otherwise} \end{array} \right\} \qquad i = 1,2,3,4$$

The 0/1 nature of each variable makes the objective function and the constraints work out as one would wish. For example, if item 1 is carried ($Y_1 = 1$) on the airplane, then it contributes 29.5 to revenue; if item 1 is not carried ($Y_1 = 0$), then it contributes nothing. Hence, whether or not item 1 is carried, it contributes $29.5Y_1$ to revenue. Applying this argument for each item the objective function is

$$\text{maximize } 29.5Y_1 + 11.3Y_2 + 34.0Y_3 + 19.7Y_4$$

If item 1 is carried on the plane, then it uses 4.3 Tonnes of the available capacity; otherwise, it uses nothing. Hence, whether or not item 1 is carried on the plane, it uses $4.3Y_1$ Tonnes of the capacity. Continuing this argument for the other three items, the weight restriction is

$$4.3Y_1 + 1.9Y_2 + 5.8Y_3 + 3.6Y_4 \leq 12.3$$

Finally, we must have:

$$Y_i \in \{0,1\} \qquad i = 1,2,3,4$$

6.3.3 Example 3

Problem descriptions often contain the words "if" or "only if". Suppose that five locations for a food franchise have been identified, and that the problem statement contains the words "if a restaurant is opened at location 3, then one will be opened at location 5 also." At each location there is binary choice and therefore we define:

$$Y_i = \left\{ \begin{array}{ll} 1 & \text{if a restaurant is opened at location } i \\ 0 & \text{otherwise} \end{array} \right\} \qquad i = 1,2,3,4,5$$

The phrase "if a restaurant is opened at location 3" is equivalent to "if $Y_3 = 1$". The full phrase is equivalent to "if $Y_3 = 1$ then $Y_5 = 1$." Since each of Y_3 and Y_5 is either 0 or 1, the requirement is met by the constraint $Y_5 \geq Y_3$. If $Y_3 = 0$, then we obtain $Y_5 \geq 0$ which is trivially true; if $Y_3 = 1$ then the constraint forces Y_5 to be greater than or equal to 1, but since Y_5 must be either 0 or 1, the combined effect

[5] In this chapter we reserve the letter Y for representing 0/1 variables.

is to force Y_5 to be *exactly* one. The logically correct expression $Y_5 \geq Y_3$ can be re-written as either

$$-Y_3 + Y_5 \geq 0$$

or as

$$Y_3 - Y_5 \leq 0$$

The statement "a restaurant could be opened at location 1 only if a restaurant is opened at location 4" requires a relationship between Y_1 and Y_4. The logic requires Y_1 to be 0 if Y_4 is 0; if $Y_4 = 1$, then Y_1 can be either 0 or 1. Hence we must have $Y_1 \leq Y_4$, or equivalently

$$Y_1 - Y_4 \leq 0$$

A statement in words may require more than one constraint. For example, "a restaurant must be opened at location 2 if one is opened at either location 3 or 4" implies that $Y_2 \geq Y_3$ and $Y_2 \geq Y_4$, i.e.

$$\begin{array}{rcl} Y_2 - Y_3 & \geq & 0 \\ Y_2 - Y_4 & \geq & 0 \end{array}$$

On the other hand, the statement "a restaurant must be opened at location 2 if one is opened at both locations 3 and 4" logically requires Y_2 to be 1 if the sum of Y_3 and Y_4 is 2. Hence $Y_2 \geq Y_3 + Y_4 - 1$, or equivalently

$$-Y_2 + Y_3 + Y_4 \leq 1$$

Finally, we must have:

$$Y_i \in \{0,1\} \qquad i = 1,2,3,4,5$$

6.3.4 Example 4

A US based firm wishes to export to the European Union. One of its options to accomplish this is the building of a warehouse in Rotterdam. This binary choice is conveniently modelled using a 0/1 variable:

$$Y_1 = \begin{cases} 1 & \text{if a warehouse is built in Rotterdam} \\ 0 & \text{otherwise} \end{cases}$$

Suppose that the warehouse would cost seven million US dollars to build. If the units of the (minimization) objective function are in millions of US dollars, then a $7Y_1$ would appear in the objective function.[6] If the warehouse is built ($Y_1 = 1$), then a cost of $7 \times 1 = 7$ units is incurred; if the warehouse is not built ($Y_1 = 0$), then a cost of $7 \times 0 = 0$ is incurred.

Suppose that the warehouse, if built, would have a capacity of 42,000 cubic metres. Let the volume (in cubic metres) of the occupied space in the warehouse be modelled by X_1. Therefore we require that:

$$X_1 \leq 42000$$

If the warehouse is not built, then $X_1 = 0$. We can force this to be the case by using a constraint such as:

$$X_1 \leq 90000Y_1$$

The number by which Y_1 is multiplied can be anything that's at least 42,000. Since $X_1 \geq 0$ will be one of the non-negativity restrictions, the two constraints taken together will enforce the following:

$$\begin{array}{lcll} X_1 & \leq & 42000 & \text{if } Y_1 = 1 \\ X_1 & = & 0 & \text{if } Y_1 = 0 \end{array}$$

However, we can collapse $X_1 \leq 42000$ and $X_1 \leq 90000Y_1$ into one by writing:

$$X_1 \leq 42000Y_1$$

Keeping the convention that all variables must appear on the left we can re-write this as

$$X_1 - 42000Y_1 \leq 0$$

Finally, at the end we write $X_1 \geq 0$, and $Y_1 \in \{0, 1\}$.

6.3.5 Example 5

An assembly line costs \$1000 to set up. Once set up, each unit produced contributes \$3.70 to profit. The line has a capacity of 800 units before it needs to be re-set. If we let X represent the number of units produced, then the contribution to profit is:

$$\begin{array}{rl} 3.7X - 1000 & \text{if } X > 0 \\ 0 & \text{if } X = 0 \end{array}$$

[6] If this were a maximization model then a $-7Y_1$ term would appear.

Here, an important managerial decision is whether or not to set up the line, which we can represent using the variable Y.

$$Y = \begin{cases} 1 & \text{if the line is set up} \\ 0 & \text{otherwise} \end{cases}$$

The contribution to profit from this operation is a function of both variables X and Y.

$$\text{contribution } = 3.7X - 1000Y$$

We require that X not exceed the maximum production on the line before it needs to be re-set, i.e. $X \leq 800$. Also, we need to require that X be 0 when Y is 0. We can accomplish this by a constraint such as $X \leq 1000Y$. In this example, we can create one constraint which acts as both a capacity constraint and a logical relationship constraint, simply by writing:

$$X \leq 800Y$$

This inequality, combined with the non-negativity restriction $X \geq 0$, means that $0 \leq X \leq 800$ when $Y = 1$, and $X = 0$ when $Y = 0$. Putting the Y variable on the left we have:

$$X - 800Y \leq 0$$

Finally, we write $X \geq 0$, and $Y \in \{0,1\}$.

6.3.6 Example 6

The two previous examples have given capacity restrictions. When no capacity restriction is given, a parameter "M" must be introduced. Suppose that an input to a firm costs \$2.30 per kg, based on a minimum order quantity of 100 kgs. If X represents the amount (in kgs) purchased, then either $X = 0$, or $X \geq 100$. In a minimization objective function there is, as one would expect, a $2.3X$ term, but two more constraints are needed to handle the discontinuity in X. First, we need to define a 0/1 variable Y:

$$Y = \begin{cases} 1 & \text{if } X \geq 100 \\ 0 & \text{if } X = 0 \end{cases}$$

(Note that the use of the word *otherwise* would not have been applicable here.) By writing the constraint $X \geq 100Y$, or equivalently,[7]

$$-X + 100Y \leq 0$$

X is forced to be at least 100 if $Y = 1$. But this is not enough. We also need to force X to be 0 if $Y = 0$. This is accomplished in part by the non-negativity restriction $X \geq 0$, and in part by a constraint of the form

$$X \leq MY$$

or

$$X - MY \leq 0$$

Here, "M" represents a large number, at least as large as X is likely to be. Suppose that X would certainly be no more than 4000 kgs. Then the constraint $X - 4000Y \leq 0$ forces X to be 0 if $Y = 0$, and allows X to be as much as 4000 if $Y = 1$. The "M" is used in a formulation to indicate the logic of the model. When *solving* the model on a computer, a particular value for M must be used. The particular value for M should be as large as is needed, but given this, should be as small as possible for the sake of computational efficiency. As before, at the end we add $X \geq 0$, and $Y \in \{0, 1\}$.

6.3.7 Example 7 (Optional)

Normally, each constraint of a model must be satisfied. Sometimes, however, there is a pair of constraints such that at least one (but not necessarily both) must be satisfied. For example, suppose that we require that either

$$4X_1 + 7X_2 \leq 25 \qquad (1)$$

or

$$5X_1 + 3X_2 \leq 32 \qquad (2)$$

but we do not *require* that both be satisfied. To handle this situation we define:

$$Y = \begin{cases} 1 & \text{if constraint (2) must be satisfied} \\ 0 & \text{if constraint (1) must be satisfied} \end{cases}$$

[7] The form $-X + 100Y \leq 0$ is more computationally efficient than $X - 100Y \geq 0$. This is because the simplex algorithm needs to add what is called an *artificial variable* for each $\geq$ (and each =) constraint.

As in Example 6, we use a suitably large number "M". We now alter the constraints to obtain:

$$\begin{aligned} 4X_1 + 7X_2 &\leq 25 + MY \\ 5X_1 + 3X_2 &\leq 32 + M(1 - Y) \end{aligned}$$

If $Y = 0$, then we have

$$\begin{aligned} 4X_1 + 7X_2 &\leq 25 \\ 5X_1 + 3X_2 &\leq 32 + M \end{aligned}$$

Since M is large, the second constraint becomes redundant, yielding

$$4X_1 + 7X_2 \leq 25$$

If $Y = 1$, then we have

$$\begin{aligned} 4X_1 + 7X_2 &\leq 25 + M \\ 5X_1 + 3X_2 &\leq 32 \end{aligned}$$

which is equivalent, since M is large, to

$$5X_1 + 3X_2 \leq 32$$

As before, M needs to be numerically specified in order to solve the model using a computer. If M is 1000, then we obtain

$$\begin{aligned} 4X_1 + 7X_2 - 1000Y &\leq 25 \\ 5X_1 + 3X_2 + 1000Y &\leq 1032 \end{aligned}$$

Finally, we require that $X_1 \geq 0$, $X_2 \geq 0$, $Y \in \{0, 1\}$.

Exercise

Suppose now that we were to change both constraints of Example 7 to be *equality* constraints. Show how to formulate this modified model.

6.4 Formulation Problems with 0/1 Variables

6.4.1 Locating Distribution Terminals

Description

Suppose that a company based in St. John's is considering adding distribution terminals in (1) Halifax, (2) Moncton, (3) Montréal, (4) Ottawa, and (5) Toronto. The cost of building the five terminals in millions of dollars would be 10 in Halifax, 12 in Moncton, 20 in Montréal, 18 in Ottawa, and 25 in Toronto. No more than two terminals may be built. If no terminal is built in Halifax, then one must be built in Moncton. At least one terminal must be built in central Canada (i.e. at locations (3), (4), or (5)).

Formulation

In each of these cities they either build the terminal or do not, so we can let a subscripted variable handle each decision:

$$Y_1 = \begin{cases} 1 & \text{if a terminal is built in city 1 (Halifax)} \\ 0 & \text{otherwise} \end{cases}$$

$$Y_2 = \begin{cases} 1 & \text{if a terminal is built in city 2 (Moncton)} \\ 0 & \text{otherwise} \end{cases}$$

and so on. Instead of defining five variables separately, we could define all five at once:

$$Y_i = \begin{cases} 1 & \text{if a terminal is built in city } i \ \ (i = 1, \ldots, 5) \\ 0 & \text{otherwise} \end{cases}$$

In writing the objective function, the beauty of the 0/1 nature of the variables becomes apparent, because the cost of building in Halifax is $10Y_1$; if they build there the cost would be 10(1) = 10, and if they don't build there the cost will be 10(0) = 0. Hence the cost of building the terminals will be:

$$10Y_1 + 12Y_2 + 20Y_3 + 18Y_4 + 25Y_5$$

There is a constraint that no more than two terminals may be built; this is simply:

$$Y_1 + Y_2 + Y_3 + Y_4 + Y_5 \leq 2$$

We need a constraint to ensure that if no terminal is built in Halifax, then one must be built in Moncton. What we are saying is that "if $Y_1 = 0$, then $Y_2 = 1$". We see that this can be accomplished by (1) building in Halifax alone; (2) building in Moncton alone; or (3) building in both Halifax and Moncton. The way to write this is:

$$Y_1 + Y_2 \geq 1$$

Finally, we need at least one terminal built in central Canada (i.e. one of Montréal, Ottawa, or Toronto). This is:

$$Y_3 + Y_4 + Y_5 \geq 1$$

At the end of the formulation we simply write that these variables must be 0/1. For the Solver on Excel, we would use the "Add constraint" feature to declare the variable cells to be "bin" (meaning "binary").

$$\begin{array}{rrcl}
\text{minimize} & 10Y_1 + 12Y_2 + 20Y_3 + 18Y_4 + 25Y_5 & & \\
\text{subject to} & & & \\
\text{at most two terminals} & Y_1 + Y_2 + Y_3 + Y_4 + Y_5 & \leq & 2 \\
\text{Halifax or Moncton} & Y_1 + Y_2 & \geq & 1 \\
\text{central Canada} & Y_3 + Y_4 + Y_5 & \geq & 1 \\
 & & & \\
 & Y_i \in \{0,1\} \;\; i = 1, \ldots, 5 & &
\end{array}$$

This little example is so simple that we can solve it just by looking at it. The optimal solution is to build terminals in Halifax and Ottawa for a total cost of 28 million dollars.

6.4.2 Buying Family Pets

Description

John and Janet Noseworthy have three children named Becky, Peter, and Alice. They have mentioned the idea of buying some pets, and the children are delighted. Becky would like a cat, a big dog, and a bird; Peter wants a cat, a little dog, and a big dog; and Alice would like a little dog, a big dog, and an aquarium of fish. The cost to purchase and look after these animals for a year would be:

Pet	Cat	Little Dog	Big Dog	Bird	Fish
Cost	\$1000	\$1300	\$1800	\$400	\$600

They will only buy one of any kind of pet. For example, if they buy a cat, Becky and Peter will share him/her. The parents have promised that each child will receive at least two of his/her wishes. They have a budget of \$3600. They won't buy both a little dog and a big dog. They won't buy both a cat and a bird. They wish to maximize the number of each child's wishes granted.

Formulation

Let the five animals be indexed from 1 to 5 as follows: 1 cat; 2 little dog; 3 big dog; 4 bird; 5 fish.

$$Y_i = \begin{cases} 1 & \text{if animal } i \text{ is purchased} \quad (i = 1, \ldots, 5) \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{array}{rrcl}
\text{maximize} & 2Y_1 + 2Y_2 + 3Y_3 + Y_4 + Y_5 & & \\
\text{subject to} & & & \\
\text{Becky's wishes} & Y_1 + Y_3 + Y_4 & \geq & 2 \\
\text{Peter's wishes} & Y_1 + Y_2 + Y_3 & \geq & 2 \\
\text{Alice's wishes} & Y_2 + Y_3 + Y_5 & \geq & 2 \\
\text{Budget} & 1000Y_1 + 1300Y_2 + 1800Y_3 + & & \\
 & 400Y_4 + 600Y_5 & \leq & 3600 \\
\text{not both dogs} & Y_2 + Y_3 & \leq & 1 \\
\text{not a cat and a bird} & Y_1 + Y_4 & \leq & 1
\end{array}$$

$$Y_i \in \{0, 1\} \quad i = 1, \ldots, 5$$

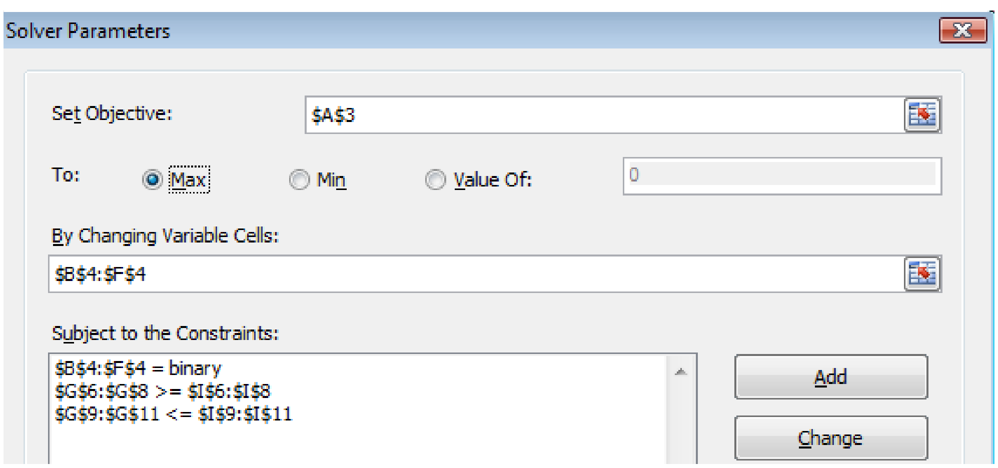
Solution

This is put onto a spreadsheet, with `=SUMPRODUCT(B3:F3,B4:F4)` in cell A3, and `=SUMPRODUCT($B$4:$F$4,B6:F6)` in cell G6, which is copied into the range G6:G11.[8]

[8] Note that we did not put the adjectives "Little" and "Big" in their own cells C1 and D1, but instead they appear as part of the variable names in row 2. This makes columns C and D rather wide, but it ensures that the correct wording will appear in the Answer Report.

	A	B	C	D	E	F	G	H	I
1	Buying Pets								
2	Wishes granted	Cat	Little Dog	Big Dog	Bird	Fish			
3	0	2	2	3	1	1			
4									
5									
6	Becky's wishes	1		1	1		0	>=	2
7	Peter's wishes	1	1	1			0	>=	2
8	Alice's wishes		1	1		1	0	>=	2
9	Budget	1000	1300	1800	400	600	0	<=	3600
10	not both dogs		1	1			0	<=	1
11	not a cat and a bird	1			1		0	<=	1

Things are similar to that of a linear model, except that we must declare the variable cells (the range B4:F4) to be binary. Doing this the top part of the Solver Parameter box is as follows:



The model is then solved to obtain:

	A	B	C	D	E	F	G	H	I
1	Buying Pets								
2	Wishes granted	Cat	Little Dog	Big Dog	Bird	Fish			
3	6	2	2	3	1	1			
4		1	0	1	0	1			
5									
6	Becky's wishes	1		1	1		2	>=	2
7	Peter's wishes	1	1	1			2	>=	2
8	Alice's wishes		1	1		1	2	>=	2
9	Budget	1000	1300	1800	400	600	3400	<=	3600
10	not both dogs		1	1			1	<=	1
11	not a cat and a bird	1			1		1	<=	1

We see that they buy a cat, a big dog, and a fish, and that six of the children's wishes are granted. The top part of the Answer Report is:

Objective Cell (Max)

Cell	Name	Original Value	Final Value
$A$3	Wishes granted	0	6

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$4	Cat	0	1	Binary
$C$4	Little Dog	0	0	Binary
$D$4	Big Dog	0	1	Binary
$E$4	Bird	0	0	Binary
$F$4	Fish	0	1	Binary

The variables which are in the solution are the ones which are 1 in the Final Value column, i.e. a cat, a big dog, and an aquarium of fish.

6.4.3 A Covering Problem

The specific application discussed here is fire protection, but the context puts it into the general problem of *covering*. In an earlier chapter, we saw an example of covering where the specific application was the determination of the requirements for police constables.

Description

The Avalon Regional Government has six sectors which need fire protection. Adequate fire protection can be provided in each sector either by building a fire station in that sector, or by building a fire station in another sector which is no more than a 12 minute drive away. The time to drive between the centres of each pair of sectors is given in the following table. (Because of one-way streets and left-turns the times are not symmetric.) The cost to build a fire station is the same in each sector. We wish to formulate a model whose purpose is to choose which sectors should have their own fire station.

	To					
From	1	2	3	4	5	6
1	0	7	15	21	23	18
2	9	0	17	20	18	11
3	13	18	0	12	8	19
4	18	14	20	0	28	10
5	13	10	12	14	0	23
6	19	13	7	16	8	0

Formulation

The problem as stated has a straightforward formulation. In each sector, either a fire station is built, or it is not built. Hence we define:

$$Y_i = \left\{ \begin{array}{ll} 1 & \text{if a fire station is built in sector } i \\ 0 & \text{otherwise} \end{array} \right\} \qquad i = 1,2,3,4,5,6$$

Consider sector 1. It can be reached from sector 2 in 9 (≤ 12) minutes, but it cannot be reached from any other sector in the 12 minute window. Therefore, in order to provide adequate fire protection to sector 1, a station must be built in either sector 1 or sector 2 (or both). Hence we require that

$$Y_1 + Y_2 \geq 1$$

Sector 2 can be reached within 12 minutes from either sector 1 (7 minutes) or sector 5 (10 minutes). Hence we require that

$$Y_1 + Y_2 + Y_5 \geq 1$$

We continue this process for the other four sectors, in each case finding the times in the *column* which do not exceed 12. Since the cost of building a fire station is the same in each sector, we can minimize the total cost by minimizing the number of fire stations built. Hence the model is

$$\begin{array}{lcccccccccccccc}
\text{minimize} & Y_1 & + & Y_2 & + & Y_3 & + & Y_4 & + & Y_5 & + & Y_6 & & \\
\text{subject to} & & & & & & & & & & & & & \\
\text{Sector 1} & Y_1 & + & Y_2 & & & & & & & & & \geq & 1 \\
\text{Sector 2} & Y_1 & + & Y_2 & & & & & + & Y_5 & & & \geq & 1 \\
\text{Sector 3} & & & & & Y_3 & & & + & Y_5 & + & Y_6 & \geq & 1 \\
\text{Sector 4} & & & & & Y_3 & + & Y_4 & & & & & \geq & 1 \\
\text{Sector 5} & & & & & Y_3 & & & + & Y_5 & + & Y_6 & \geq & 1 \\
\text{Sector 6} & & & Y_2 & & & + & Y_4 & & & + & Y_6 & \geq & 1
\end{array}$$

$$Y_i \in \{0,1\} \quad (i = 1,\ldots,6)$$

A More Complicated Model (Optional)

If the original statement of the problem had not had specific numbers, but instead the times from sector i to sector j had been given as t_{ij}, then a more complex formulation would have resulted. Of course, this more complex model is more robust.

To handle this problem, we need an additional set of variables. [9] We define

$$F_{ij} = \left\{ \begin{array}{ll} 1 & \text{if sector } i \text{ has a station} \\ & \text{serving sector } j \\ 0 & \text{otherwise} \end{array} \right\} \quad i = 1,\ldots,6 \quad j = 1,\ldots,6$$

As before, the objective function is

$$\text{minimize} \sum_{i=1}^{6} Y_i$$

Each sector j needs to be served from somewhere, therefore we obtain the following six constraints:

$$\sum_{i=1}^{6} F_{ij} \geq 1 \qquad j = 1,\ldots,6$$

[9] The additional set is required; the original six variables do not have to be used, but it is easier to do so.

Sector i can only serve j if the travel time does not exceed 12 minutes. This gives 30 constraints:

$$t_{ij}F_{ij} \leq 12 \qquad i = 1,\ldots,6 \quad j = 1,\ldots,6 \quad j \neq i$$

Sector i can only serve sector j if sector i has a fire station. Since both F_{ij} and Y_i are 0/1 variables, we can accomplish this by writing, for each i and j, $F_{ij} \leq Y_i$, or equivalently, $F_{ij} - Y_i \leq 0$. (In the case where $i = j$, we could write $F_{ii} - Y_i = 0$, rather than $F_{ii} - Y_i \leq 0$, without affecting the solution.) The complete formulation is:

$$\begin{array}{lrcll}
\text{minimize} & \sum_{i=1}^{6} Y_i & & & \\
\text{subject to} & & & & \\
(1)\ldots(6) & \sum_{i=1}^{6} F_{ij} & \geq & 1 & (j = 1,\ldots,6) \\
(7)\ldots(36) & t_{ij}F_{ij} & \leq & 12 & \left\{ \begin{array}{l} i = 1,\ldots,6 \\ j = 1,\ldots,6 \\ j \neq i \end{array} \right\} \\
(37)\ldots(72) & F_{ij} - Y_i & \leq & 0 & \left\{ \begin{array}{l} i = 1,\ldots,6 \\ j = 1,\ldots,6 \end{array} \right\} \\
& Y_i, F_{ij} \in \{0,1\} & & & \left\{ \begin{array}{l} i = 1,\ldots,6 \\ j = 1,\ldots,6 \end{array} \right\}
\end{array}$$

6.4.4 A Fixed Charge Problem

When a problem contains a cost of the all-or-nothing type, we have a *fixed charge* problem. An example of this is a water utility which charges \$400 per annum for hookup to the water mains regardless of consumption, rather than equip each consumer with a meter so that they can be charged at, for example, \$0.002 per litre. Sometimes, a product has both a fixed and a variable component, for example, an power utility might charge a connection fee of \$17 per month and a consumption fee of \$0.11 per kilowatt-hour. Often a product is sold as if the cost were variable but in reality it is essentially fixed. An example of such a product is airline seats; it does not cost much more to fly a plane which is nearly full than one which is nearly empty. Hence, an airline which is trying to model its operations would probably represent each flight as a fixed charge which can be avoided only by canceling the flight. Each fixed charge needs to be modeled using a 0/1 variable, as the following example illustrates.

Description

A firm wishes to produce a single product at one or more locations so that the total monthly cost is minimized subject to demand being satisfied. At each location there is a fixed charge to be paid if any are produced (but is nil otherwise), and a variable cost which depends on whether the units are produced on regular time or on overtime. Each location has capacity restrictions on regular and overtime production. The relevant data are:

Plant	Fixed	Regular Time		Overtime	
Location	Cost	Unit Cost	Capacity	Unit Cost	Capacity
1	2000	3.80	1000	4.60	400
2	3000	2.90	1200	4.10	550
3	1500	4.20	1500	5.60	600
4	2400	3.40	1300	4.20	450
5	2700	3.60	1400	5.10	500

Demand is for 5100 units per month.

Formulation

Whenever there is a cost which is either 0 or a fixed amount greater than 0, depending on whether the production level is 0 or greater than 0, a 0/1 variable is needed to model the situation. We define

$$Y_i = \left\{ \begin{array}{ll} 1 & \text{if the production at location } i \text{ is} > 0 \\ 0 & \text{otherwise} \end{array} \right\} \qquad i = 1,2,3,4,5$$

Production at each location may occur on regular time, or on both regular time and overtime. We define:

$$\begin{aligned} R_i &= \text{regular time production level at location } i \quad i = 1,2,3,4,5 \\ O_i &= \text{overtime production level at location } i \quad i = 1,2,3,4,5 \end{aligned}$$

The objective function is therefore:

$$\begin{aligned} \text{minimize} \quad & 2000Y_1 + 3000Y_2 + 1500Y_3 + 2400Y_4 + 2700Y_5 \\ + \quad & 3.8R_1 + 2.9R_2 + 4.2R_3 + 3.4R_4 + 3.6R_5 \\ + \quad & 4.6O_1 + 4.1O_2 + 5.6O_3 + 4.2O_4 + 5.1O_5 \end{aligned}$$

The capacity of regular time production at plant 1 is 1000 units. Hence we require that:

$$R_1 \leq 1000$$

Also, we must force R_1 to be 0 if $Y_1 = 0$, This logical relationship is ensured by having a constraint of the form $R_1 \leq MY_1$, where M is any sufficiently large number.[10] We could require, for example, that:

$$R_1 \leq 5000Y_1$$

Hence we limit the capacity by writing $R_1 \leq 1000$, and we enforce the requirement that the production be 0 if the plant is closed, by writing $R_1 \leq 5000Y_1$, or equivalently $R_1 - 5000Y_1 \leq 0$. Later, we present the entire formulation in which this conceptualization is used. For all five plants we have:

$$\begin{aligned} R_1 &\leq 1000 \\ R_2 &\leq 1200 \\ R_3 &\leq 1500 \\ R_4 &\leq 1300 \\ R_5 &\leq 1400 \end{aligned}$$

At each plant i, there can be no regular time production if the plant is closed. The value of $M = 5000$ can be used for all inequalities:

$$R_i - 5000Y_i \leq 0 \ (i = 1,2,3,4,5)$$

In a situation like this, we can accomplish both the need to limit the capacity and the requirement to make the production 0 when the plant is closed simply by using one constraint:

$$R_1 \leq 1000Y_1$$

This single inequality is valid because if $Y_1 = 1$, we create the constraint $R_1 \leq 1000$, and if $Y_1 = 0$, we force R_1 to be 0. This constraint can be re-written as:

$$R_1 - 1000Y_1 \leq 0$$

[10] For this particular constraint, any number that's at least 1000 would work.

Doing this for all locations we obtain:

$$\begin{aligned} R_1 - 1000Y_1 &\leq 0 \\ R_2 - 1200Y_2 &\leq 0 \\ R_3 - 1500Y_3 &\leq 0 \\ R_4 - 1300Y_4 &\leq 0 \\ R_5 - 1400Y_5 &\leq 0 \end{aligned}$$

We later present a model which uses the above conceptualization, which reduces the number of constraints.

A similar set of constraints applies to the overtime production. Using the first way we write:

$$\begin{aligned} O_1 &\leq 400 \\ O_2 &\leq 550 \\ O_3 &\leq 600 \\ O_4 &\leq 450 \\ O_5 &\leq 500 \end{aligned}$$

At each plant i, there can be no overtime production if the plant is closed:

$$O_i - 5000Y_i \leq 0 \ (i = 1,2,3,4,5)$$

Using the second way in which these constraints are combined, we have:

$$\begin{aligned} O_1 - 400Y_1 &\leq 0 \\ O_2 - 550Y_2 &\leq 0 \\ O_3 - 600Y_3 &\leq 0 \\ O_4 - 450Y_4 &\leq 0 \\ O_5 - 500Y_5 &\leq 0 \end{aligned}$$

Although logically O_i must be 0 unless R_i is at its capacity, there is no need to force this logic by using constraints. This is because the objective is to minimize cost, and hence at any location the cheaper regular time production would have to be at its capacity before any of the more expensive overtime production could begin.[11]

We must ensure that the demand is met:

$$\sum_{i=1}^{5}(R_i + O_i) \geq 5100$$

Finally, we note that each Y_i variable must be $\in \{0,1\}, i = 1,2,3,4,5$, but each R_i and each O_i merely has the usual non-negativity restriction.[12]

In summary, the variables are:

$$\begin{aligned}
Y_i &= \left\{ \begin{array}{ll} 1 & \text{if the production at location } i \text{ is} > 0 \\ 0 & \text{otherwise} \end{array} \right\} \quad i = 1,2,3,4,5 \\
R_i &= \text{regular time production level at location } i \quad i = 1,2,3,4,5 \\
O_i &= \text{overtime production level at location } i \quad i = 1,2,3,4,5
\end{aligned}$$

The model is now presented in two versions. The first contains ten more constraints than the second one, caused by separating the capacity constraints from the constraints which force the production to be nil if the plant is closed.

---

[11] If, for some reason, overtime were cheaper than regular time, then of course a different model would result.

[12] This is true even if the units are integral things such as screwdrivers rather than continuous things such as litres of paint. The presence of integral demand and capacities combined with the seeking of corner point solutions by the simplex algorithm will ensure that each R_i and each O_i is integer, hence there is no need to make these variables explicitly integer. Even if this were not so, the integrality of these variables is unimportant compared with the fixed charge variables.

$$
\begin{array}{llll}
\text{minimize} & \multicolumn{3}{l}{2000Y_1 + 3000Y_2 + 1500Y_3 + 2400Y_4 + 2700Y_5} \\
& \multicolumn{3}{l}{+\ 3.8R_1 + 2.9R_2 + 4.2R_3 + 3.4R_4 + 3.6R_5} \\
& \multicolumn{3}{l}{+\ 4.6O_1 + 4.1O_2 + 5.6O_3 + 4.2O_4 + 5.1O_5} \\
\multicolumn{4}{l}{\text{subject to}} \\
\multicolumn{4}{l}{\text{Capacity on Regular Time at Plant}} \\
1 & R_1 & \leq & 1000 \\
2 & R_2 & \leq & 1200 \\
3 & R_3 & \leq & 1500 \\
4 & R_4 & \leq & 1300 \\
5 & R_5 & \leq & 1400 \\
\multicolumn{4}{l}{\text{No Regular Time if Plant } i \text{ is Closed}} \\
\multicolumn{4}{l}{\text{Big M Method with M = 5000}} \\
& R_i - 5000Y_i & \leq & 0 \ (i = 1,2,3,4,5) \\
\multicolumn{4}{l}{\text{Capacity on Overtime at Plant}} \\
1 & O_1 & \leq & 400 \\
2 & O_2 & \leq & 550 \\
3 & O_3 & \leq & 600 \\
4 & O_4 & \leq & 450 \\
5 & O_5 & \leq & 500 \\
\multicolumn{4}{l}{\text{No Overtime if Plant } i \text{ is Closed}} \\
\multicolumn{4}{l}{\text{Big M Method with M = 5000}} \\
& O_i - 5000Y_i & \leq & 0 \ (i = 1,2,3,4,5) \\
\text{Demand} & \sum_{i=1}^{5}(R_i + O_i) & \geq & 5100 \\
& & & \\
& \multicolumn{3}{l}{\text{all variables } \geq 0} \\
& \multicolumn{3}{l}{Y_i \in \{0,1\},\ i = 1,2,3,4,5}
\end{array}
$$

In the second version, the size is reduced by imbedding the capacity within the constraints that enforce that there be no production when the plant is closed.

$$
\begin{array}{rlcl}
\text{minimize} & 2000Y_1+3000Y_2+1500Y_3+2400Y_4+2700Y_5 & & \\
& +\ 3.8R_1+2.9R_2+4.2R_3+3.4R_4+3.6R_5 & & \\
& +\ 4.6O_1+4.1O_2+5.6O_3+4.2O_4+5.1O_5 & & \\
\multicolumn{4}{l}{\text{subject to}} \\
\multicolumn{4}{l}{\text{Capacity on Regular Time at Plant}} \\
1 & R_1-1000Y_1 & \le & 0 \\
2 & R_2-1200Y_2 & \le & 0 \\
3 & R_3-1500Y_3 & \le & 0 \\
4 & R_4-1300Y_4 & \le & 0 \\
5 & R_5-1400Y_5 & \le & 0 \\
\multicolumn{4}{l}{\text{Capacity on Overtime at Plant}} \\
1 & O_1-400Y_1 & \le & 0 \\
2 & O_2-550Y_2 & \le & 0 \\
3 & O_3-600Y_3 & \le & 0 \\
4 & O_4-450Y_4 & \le & 0 \\
5 & O_5-500Y_5 & \le & 0 \\
\text{Demand} & \sum_{i=1}^{5}(R_i+O_i) & \ge & 5100 \\
& & & \\
& \text{all variables } \ge 0 & & \\
& Y_i \in \{0,1\},\ i=1,2,3,4,5 & &
\end{array}
$$

6.4.5 A Model with Economies of Scale

Description

A firm purchases an input from a supplier. The unit price of this input depends on the quantity ordered:

Range	Unit Cost
First 100 units	\$6.90
Next 400 units	\$5.10
Each additional unit	\$3.70

This type of cost structure encourages large infrequent orders in order to obtain a low average cost per unit. For example, to purchase 800 units would cost 100@\$6.90 + 400@\$5.10 + 300@\$3.70 = \$3840 or \$4.80 each.

We are now at 30 June, with an inventory of 260 units. The production plan requires the following number of units over the next six months:

July	August	September	October	November	December
700	650	380	900	320	450

There is a charge of \$0.80 to hold one unit in inventory for one month; this charge is based on the ending inventory in each month. We desire an end-of-year inventory of at least 400 units. We wish to formulate a model which will minimize the sum of purchase and inventory costs.

Formulation

As with the inventory problems which we saw in an earlier chapter, we define

$$I_t = \text{ the inventory level at the end of month } t, \quad t = 0, \ldots, 6$$

Were it not for the varying per unit cost, we would have defined X_t as the amount purchased in month t. Because, however, there are three prices which they could pay (where 1 is \$6.90 and 3 is \$3.70), it is tempting to define X_{ti} as the amount purchased in month t at price level i. This would be fine if the unit cost *increased* (or stayed the same) as a function of quantity ordered. An increase in cost would be like the previous example in which overtime, which would only be used once the regular time capacity has been reached, costs more per-unit than regular time. However, in this example, the unit cost is decreasing, hence we need a new way of looking at the problem.

Going back to X_t, we break the function up into its three parts:

1. If $X_t \leq 100$, then the cost is $6.9X_t$.
2. Since the cost of 100 units is 100@\$6.90 = \$690, then if $100 \leq X_t \leq 500$, the cost is
$$690 + 5.1(X_t - 100) = 180 + 5.1X_t.$$
3. Since the cost of 500 units is \$690 + 400@\$5.10 = 2730, if at least 500 units are purchased then the cost is
$$2730 + 3.7(X_t - 500) = 880 + 3.7X_t.$$

To put all this into an objective function we will need to use three separate X-type variables, which brings us back to X_{ti}. However, we see that the definition which we need is not the number sold at each price level, but instead

$$\begin{aligned} X_{ti} \quad = \quad & \text{amount purchased in month } t \text{ where the last unit} \\ & \text{is sold at price level } i, \quad t = 1, \ldots, 6 \quad i = 1, 2, 3 \end{aligned}$$

Hence, for each t, only one of the three X_{ti} variables can be strictly positive. In the formulation, we will need to enforce this logic using 0/1 variables. We define

$$Y_{ti} = \left\{ \begin{array}{ll} 1 & \text{if } X_{ti} > 0 \\ 0 & \text{otherwise} \end{array} \right\} \qquad t = 1, \ldots, 6 \quad i = 2, 3$$

In month t, the purchase cost is:

$$6.9X_{t1} + 180Y_{t2} + 5.1X_{t2} + 880Y_{t3} + 3.7X_{t3}$$

In month t, the inventory cost is $0.8I_t$. Since the initial inventory level is *fixed* at $I_0 = 260$, the term $0.8I_0$ may be included or excluded according to the decision maker's preference. Excluding it gives the following objective function:

$$\text{minimize} \qquad \sum_{t=1}^{6} (6.9X_{t1} + 180Y_{t2} + 5.1X_{t2} + 880Y_{t3} + 3.7X_{t3} + 0.8I_t)$$

In each month, the initial inventory (which is the previous month's ending inventory), plus the amount purchased, must equal the amount used by the production process plus the ending inventory. Expressing this algebraically with all variables on the left, and including the initial and final conditions we have:

$$\begin{array}{rcl} I_0 & = & 260 \\ I_0 + X_{1,1} + X_{1,2} + X_{1,3} - I_1 & = & 700 \\ I_1 + X_{2,1} + X_{2,2} + X_{2,3} - I_2 & = & 650 \\ I_2 + X_{3,1} + X_{3,2} + X_{3,3} - I_3 & = & 380 \\ I_3 + X_{4,1} + X_{4,2} + X_{4,3} - I_4 & = & 900 \\ I_4 + X_{5,1} + X_{5,2} + X_{5,3} - I_5 & = & 320 \\ I_5 + X_{6,1} + X_{6,2} + X_{6,3} - I_6 & = & 450 \\ I_6 & \geq & 400 \end{array}$$

Next, there are the constraints on X_{t1}.

$$\begin{array}{rcl} X_{1,1} & \leq & 100 \\ X_{2,1} & \leq & 100 \\ X_{3,1} & \leq & 100 \\ X_{4,1} & \leq & 100 \\ X_{5,1} & \leq & 100 \\ X_{6,1} & \leq & 100 \end{array}$$

We require X_{t2} to be 0 if Y_{t2} is 0, and to be between 100 and 500 if Y_{t2} is 1. The logic is captured by:

$$100Y_{t,2} \le X_{t,2} \le 500Y_{t,2} \qquad\qquad t = 1,\ldots,6$$

From this we obtain two constraints for each X_{t2} variable. One is $X_{t,2} \le 500Y_{t,2}$, or $X_{t,2} - 500Y_{t,2} \le 0$. The other is $100Y_{t,2} \le X_{t,2}$, or $-X_{t,2} + 100Y_{t,2} \le 0$. Writing these for all months we obtain:

$$\begin{array}{rcl}
X_{1,2} - 500Y_{1,2} & \le & 0 \\
X_{2,2} - 500Y_{2,2} & \le & 0 \\
X_{3,2} - 500Y_{3,2} & \le & 0 \\
X_{4,2} - 500Y_{4,2} & \le & 0 \\
X_{5,2} - 500Y_{5,2} & \le & 0 \\
X_{6,2} - 500Y_{6,2} & \le & 0 \\
-X_{1,2} + 100Y_{1,2} & \le & 0 \\
-X_{2,2} + 100Y_{2,2} & \le & 0 \\
-X_{3,2} + 100Y_{3,2} & \le & 0 \\
-X_{4,2} + 100Y_{4,2} & \le & 0 \\
-X_{5,2} + 100Y_{5,2} & \le & 0 \\
-X_{6,2} + 100Y_{6,2} & \le & 0
\end{array}$$

We need something similar for the X_{t3} variables. However, there is no purchase limit on these variables. Logically, we require that

$$\begin{array}{rcl}
X_{t3} & \le & MY_{t3} \\
X_{t3} & \ge & 500Y_{t3}
\end{array}$$

We can leave the M as it is, or we can replace it with a number. Clearly, we would not purchase more than 4000 units in any month, since this exceeds the total requirement. Letting $M = 4000$ the constraints are:

$$\begin{array}{rcl}
X_{1,3} - 4000Y_{1,3} & \le & 0 \\
X_{2,3} - 4000Y_{2,3} & \le & 0 \\
X_{3,3} - 4000Y_{3,3} & \le & 0 \\
X_{4,3} - 4000Y_{4,3} & \le & 0
\end{array}$$

$$
\begin{aligned}
X_{5,3} - 4000Y_{5,3} &\leq 0 \\
X_{6,3} - 4000Y_{6,3} &\leq 0 \\
-X_{1,3} + 500Y_{1,3} &\leq 0 \\
-X_{2,3} + 500Y_{2,3} &\leq 0 \\
-X_{3,3} + 500Y_{3,3} &\leq 0 \\
-X_{4,3} + 500Y_{4,3} &\leq 0 \\
-X_{5,3} + 500Y_{5,3} &\leq 0 \\
-X_{6,3} + 500Y_{6,3} &\leq 0
\end{aligned}
$$

Optionally, we could add a set of constraints of the form $Y_{t2} + Y_{t3} \leq 1$. These constraints will be redundant because minimizing the objective function automatically ensures that these constraints will be satisfied.

Finally, each $Y_{ti} \in \{0,1\}$, where $t = 1,\ldots,6$, and $i = 2,3$; each $X_{ti} \geq 0$, where $t = 1,\ldots,6$, and $i = 1,2,3$.

Here is the complete formulation, indexing most of the constraints.

$$
\begin{aligned}
I_t &= \text{the inventory level at the end of month } t, \quad t = 0,\ldots,6 \\
X_{ti} &= \text{amount purchased in month } t \text{ where the last unit} \\
& \text{is sold at price level } i, \quad t = 1,\ldots,6 \quad i = 1,2,3 \\
Y_{ti} &= \left\{ \begin{array}{ll} 1 & \text{if } X_{ti} > 0 \\ 0 & \text{otherwise} \end{array} \right\} \quad t = 1,\ldots,6 \quad i = 2,3
\end{aligned}
$$

$$
\begin{array}{rrcll}
\text{minimize} & \multicolumn{4}{l}{\sum_{t=1}^{6}(6.9X_{t1}+180Y_{t2}+5.1X_{t2}+880Y_{t3}+3.7X_{t3}+0.8I_t)} \\
\text{subject to} & & & & \\
\text{Initial} & I_0 & = & 260 & \\
 & I_0+X_{1,1}+X_{1,2}+X_{1,3}-I_1 & = & 700 & \\
 & I_1+X_{2,1}+X_{2,2}+X_{2,3}-I_2 & = & 650 & \\
\text{Inventory} & I_2+X_{3,1}+X_{3,2}+X_{3,3}-I_3 & = & 380 & \\
\text{Balance} & I_3+X_{4,1}+X_{4,2}+X_{4,3}-I_4 & = & 900 & \\
\text{Equations} & I_4+X_{5,1}+X_{5,2}+X_{5,3}-I_5 & = & 320 & \\
 & I_5+X_{6,1}+X_{6,2}+X_{6,3}-I_6 & = & 450 & \\
\text{Final} & I_6 & \geq & 400 & \\
 & & & & \\
 & X_{t,1} & \leq & 100 & (t=1,\ldots,6) \\
\text{Relationships} & X_{t,2}-500Y_{t,2} & \leq & 0 & (t=1,\ldots,6) \\
\text{Between the} & -X_{t,2}+100Y_{t,2} & \leq & 0 & (t=1,\ldots,6) \\
X_{ti}\text{ and }Y_{ti} & X_{t,3}-4000Y_{t,3} & \leq & 0 & (t=1,\ldots,6) \\
\text{Variables} & -X_{t,3}+500Y_{t,3} & \leq & 0 & (t=1,\ldots,6)
\end{array}
$$

$$
\begin{array}{l}
X_{t,i} \geq 0 \;\; (t=1,\ldots,6,\;\; i=1,2,3) \\
Y_{t,i} \text{ is } 0/1 \;\; (t=1,\ldots,6,\;\; i=2,3)
\end{array}
$$

In addition, we could represent the internal demand for this product in month t as d_t, where d is the vector:

$$d = (700, 650, 380, 900, 320, 450)$$

This would enable us to collapse six inventory constraints to the form

$$I_{t-1} + X_{t1} + X_{t2} + X_{t3} - I_t = d_t \qquad t = 1, \ldots, 6$$

We could also shorten the constraints horizontally as well as vertically by writing

$$I_{t-1} + \sum_{i=1}^{3} X_{ti} - I_t = d_t \qquad t = 1, \ldots, 6$$

This symbolic form is highly advantageous when the model is very large. Of course, numerical form must be used on the computer, but there are algebraic modeling systems designed for this purpose.

6.4.6 Other Models (Optional)

In Appendix C two more models are discussed. A model on capacity planning begins on page 485. A model involving a journey by rail begins on page 488.

6.5 Summary

Integer optimization adds realism in modelling to the linear situation. In particular, the need to model "either-or" type situations calls for an integer formulation. However, this increase in realism requires an increase in computational complexity. When formulating such models, we must do so such that, when the integrality restrictions are relaxed, the resulting model obeys the assumptions of linear optimization. There are some situations where rounding a fractional solution may give an appropriate solution. While models with just two integer variables are easy to solve graphically, the use of an algorithm such as the branch and bound method is needed for optimally solving larger problems. Based on this algorithm, either a spreadsheet solver or a dedicated package may be used to optimally solve integer models.

6.6 Problems for Student Completion

6.6.1 Product Mix

Jennifer is making a large fruit salad for a party. She has everything she needs at home, except for pineapples and bananas. She needs 12 pineapples, and 31 bananas. She goes to a nearby fruit stand, where she finds two vendors selling bags of mixed fruit. Vendor 1 is selling bags containing two pineapples and ten bananas for \$12 per bag. Vendor 2 is selling bags containing four pineapples and five bananas for \$16 per bag. She wants to know how many bags she should buy from each vendor to meet (or exceed) the requirements for the salad, but at the least cost possible. Formulate and solve by the graphical method to determine the best integer solution.

6.6.2 Graphing Problem

A problem has been formulated as:

$$\begin{array}{rrcrcl}
\max & 2X_1 & + & 5X_2 & & \\
\text{subject to} & & & & & \\
(1) & X_1 & + & 2X_2 & \leq & 6 \\
(2) & 5X_1 & - & 3X_2 & \leq & 9 \\
(3) & -2X_1 & + & 3X_2 & \leq & 0 \\
 & & & & & \\
 & X_1 & , & X_2 & \in & \{0,1,2,3,\ldots\}
\end{array}$$

Solve this problem graphically to determine the optimal values for X_1, X_2, and the OFV.

6.6.3 Manufacturing

A sports equipment manufacturer makes three types of squash racquets: Beginner, Intermediate, and Advanced. Each racquet uses approximately the same amount of raw materials, but different amounts of labour and machine time per racquet.

Racquet Type	Labour Hours	Machine Hours	Profit
Beginner	1.0	3.0	15
Intermediate	3.2	4.0	21
Advanced	3.0	7.0	18

During the current planning period, there are 1600 labour hours and 2300 machine hours available for the production of squash racquets. At the end of the period, production will shift to tennis racquets, and therefore partially completed squash racquets would be of no value.

(a) Formulate as an integer model.

(b) Using the Solver on Excel, what is your recommendation to the sports equipment manufacturer?

6.6.4 Allocation Problem

A wealthy executive is being transferred from New York City to Tokyo. Since the Japanese use right hand drive cars (i.e. they drive on the left), the executive decides to give his five cars to his three children, who are remaining in New York.

Let

$$Y_{ij} = \begin{cases} 1 & \text{if car } i \text{ is given to child } j,\ i = 1, \ldots, 5;\ j = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

Write a constraint or set of constraints for each of the following situations:

(a) no child may receive more than three cars

(b) each child must receive at least one car

(c) cars 2 and 4 may not be given to the same child

(d) child 2 may not receive more cars than child 1

(e) the same car cannot be given to more than one child

(f) if car 5 is given to child 1, then car 3 must be given to child 1 also

(g) if car 3 is given to child 1 *or* if car 4 is given to child 2, then car 1 must be given to child 3.

6.6.5 Covering Problem Involving Banks

There are ten communities located along Highway # 2, none of which is currently served by the Bank of New Scotland. The Bank is considering opening up to three branches in the ten communities (no more than one branch per community). They believe that they will capture 20% of the market in those communities in which a branch is constructed. Furthermore, they believe that they will capture 10% of the market in any community which does not have its own branch but which is contiguous to a community (on either or both sides) which does have a branch. Each branch would have an annual overhead cost of $130,000. Each customer would give an annual contribution to profit of $25. The population of each community is:

Community	1	2	3	4	5	6	7	8	9	10
Population (000's)	1	17	4	12	30	1	9	20	10	8

(a) Formulate this problem.

(b) Solve it using the Excel Solver.

6.6.6 Fish Plant Production

Quantfisher Inc. buys fresh fish for processing. *Quantfisher* has an annual contract with its supplier to buy up to 4600 Tonnes at the following set of incremental prices:

First 1000 Tonnes (or any fraction) may be purchased @ \$0.80/kg.
Next 2000 Tonnes (or any fraction) may be purchased @ \$1.00/kg.
Next 1600 Tonnes (or any fraction) may be purchased @ \$1.20/kg.

Quantfisher's fish processing plant can operate on a one, two, or three shift per day basis. (Shift two operates only if shift one operates, and shift three operates only if shift two operates.) For each shift there is a fixed cost which exists if the shift operates, but is nil otherwise, and a variable cost per kilogram of fish processed. The costs and shift capacities are:

Shift	Fixed Cost Per Annum	Variable Cost Per Kilogram	Annual Capacity (Tonnes)
1	\$500,000	\$0.50	1700
2	\$300,000	\$0.70	1500
3	\$100,000	\$0.95	1300

If a shift operates at all, then the minimum amount processed is 1000 Tonnes per annum on that shift (i.e. the amount processed on any shift is either 0 or greater than or equal to 1000 Tonnes).

Quantfisher can sell processed fish for \$1.80 per kg. They wish to know what they should do to maximize their annual profit.

(a) Formulate this problem.

(b) Solve it using the Excel Solver.

6.6.7 Oil Storage Problem

Formulate the following problem.

A company buys oil for heating. The supplier doesn't want to deal with small orders, so there is a minimal order size of 50 litres. The price per litre is \$0.90 per litre for the first 200 litres of an order, but each additional litre in the same order costs only \$0.75 per litre.

We are now at 31 October, with 300 litres on hand. The expected number of litres that they will need over the next seven months is:

November	December	January	February	March	April	May
400	600	750	820	650	350	150

All orders arrive on the first of every month. The capacity of the tank is 1200 litres. There is a charge of \$0.05 per litre per month; this charge is based on the ending inventory in each month. They wish to have at least 100 litres on hand at the end of May.

Chapter 7

Goal Programming and Nonlinear Models

This chapter presents two new type of models. The first several sections are devoted to the topic of goal programming, in which we seek to optimize a model which has more than one objective. Then, we present the topic of optimization in which the objective function is nonlinear, or the constraints are nonlinear, or both of these are nonlinear.

7.1 Goal Programming

7.1.1 Introduction

We have assumed until now that each problem has a single objective. It may be profit maximization, or cost minimization, or the optimization of a non-monetary objective, such as minimizing pollution or maximizing energy output. We now consider the situation where we have several goals at once.

One way in which multiple goals may arise is that there are many decision makers, or there is one decision making authority, but it is responsible to many interests. Governments have this problem in that they aim to keep taxes low, must spend on many needs (education, roads, health care and so on), and must try not to operate at a deficit. Even within private enterprise, multiple goals exist because of the responsibility not just to the firm's shareholders, but to the "stakeholders" of the firm, which includes their customers, their employees, and the communities in which the company operates, as well as the shareholders.

At this point it is worthwhile to state that some problems of conflicting objectives are simply unsolvable, because they are incorrectly formulated in the first place. For example, it does no good for a city council to say to the director of the public library, "We want you to maximize free public access and minimize the required subsidy from council." What *is* do-able is an instruction such as "maximize free public access subject to a \$1,000,000 budget" or "minimize the required subsidy subject to keeping the library open at least 100 hours per week." It may be possible to solve either of these problems by a method already seen in this book, such as linear optimization. Of course, the council could ask for 100 hours of access and restrict the budget to \$1,000,000. If this is feasible, then there is no problem. If, however, this is infeasible, then we may wish to continue with one of the methods of this chapter.

We will consider the multiple objective situation where the underlying structure of the problem is in accordance with the assumptions of linear optimization. Within this context we consider goals with i) weighted priorities and ii) absolute priorities. The latter situation is often called *preemptive* goal programming. It is the objective which differs in these two methodologies; for either case we must write the goals as constraints.

7.1.2 Deviational Variables

Until this chapter, a constraint has always been a requirement which must be satisfied: should this not be possible, then the model is infeasible. In this chapter, such a constraint is called a *system* constraint. However, we now permit a different type of constraint, called a *goal* constraint, which may be violated if need be. Sometimes, these are called *hard* and *soft* constraints respectively.

In goal programming, the multiple goals are formulated as goal constraints. A typical context is that of a target. Given that we do not have to meet it exactly, we begin with an expression which contains an approximation symbol rather than an equal sign. For example, suppose that we want the expression $3X_1 + 7X_2$ to be at or near a target value of 500. Using the symbol "$\approx$" to mean "is targeted to be" we write the goal as

$$3X_1 + 7X_2 \approx 500$$

By letting D represent the deviation from the target, we can change this expression into an equality constraint:

$$3X_1 + 7X_2 = 500 + D$$

Variable D can be either positive (overachievement of the target) or negative (underachievement of the target). We need to break this variable into its positive and negative components, because there will be different costs associated with underachievement and overachievement of the target. Hence we let

$$D^+ - D^- = D$$

where $D^+ \geq 0$ and $D^- \geq 0$. Also, we want one of these to be exactly 0.[1] This occurs naturally when using the simplex algorithm, which is the underlying algorithm of the Excel Solver.

Substituting and re-arranging we obtain:

$$3X_1 + 7X_2 + D^- - D^+ = 500$$

If this is the first goal of several, then it is useful to subscript the deviational variables:

$$3X_1 + 7X_2 + D_1^- - D_1^+ = 500$$

An alternate notation is to define U_1 as the underachievement of the first goal, and O_1 as the overachievement of the first goal, giving:

$$3X_1 + 7X_2 + U_1 - O_1 = 500$$

However, the plus and minus notation is traditional, so we will continue to use it. Of course, once we put a model into Excel we can avoid using variable names altogether.

In general, when a goal is specified as a target, and when the target can be written in the form

$$\sum_{j=1}^{n} a_{ij} X_j \approx b_i$$

we will add D_i^- and subtract D_i^+ from the left hand side, and make the expression an equality, obtaining:

$$\sum_{j=1}^{n} a_{ij} X_j + D_i^- - D_i^+ = b_i$$

In the context of a target, the decision maker is trying to minimize both underachievement and overachievement. If we only wish to minimize underachievement, then the D^+ variable can be omitted, with the expression having a greater

[1]While, for example, both $70 - 0$ and $80 - 10$ will equal 70, we want the values 70 and 0 because the 80 and 10 incur more costs.

than or equal to sign.

$$\sum_{j=1}^{n} a_{ij}X_j + D_i^- \geq b_i$$

If we only wish to minimize overachievement, then the D^- variable can be omitted, with the expression having a less than or equal to sign.

$$\sum_{j=1}^{n} a_{ij}X_j - D_i^+ \leq b_i$$

Hence if the goal is one-sided, then only one of the two deviational variables is needed, with the expression being an inequality.[2]

The objective function in goal programming is always of the minimization form. In non-preemptive goal programming, each deviational variable has a coefficient in the objective function. In preemptive goal programming, the goals are ranked and are solved in descending order of importance. In the next two sections an example of each type of model is discussed.

7.2 Weighted Goals

7.2.1 Problem Description – Smelter Model

A smelter is considering accepting concentrated ore[3] from three sources. The amounts of each in millions of Tonnes per annum are represented by X_1, X_2, and X_3. The smelter, which has 1600 employees, emits about 25,000 Tonnes per annum of sulphur dioxide (SO_2).

The company's management has stated three objectives to various stakeholders, none of these being legally binding. To the shareholders, they have stated that they hope for a contribution to profit of at least $45,000,000 per annum. To the union which represents their employees, they have stated that no layoffs are planned, but at the same time, they wish to avoid needing to hire more employees. Finally, to the public at large, they have stated their intention to reduce the pollution of SO_2 to 20,000 Tonnes per annum.

[2] Alternatively, we can use both types of variables for each goal, the expression being an equality. In the latter case, the unneeded variable will not appear in the objective function.

[3] Before ore from a mine goes to a smelter it is first crushed, then grinded, and then goes through a flotation process which separates the ore from the host rock. These three operations are called *concentration*.

At a meeting of the board of directors, several statements were made which indicate how the board views the relative cost of not meeting their goals:

1. failing to meet the profit objective by \$1,000,000 is twice as bad as exceeding the target for the SO_2 emissions by 1000 Tonnes

2. we are indifferent between a shortfall of \$1,000,000 and a underachievement in employment of 125 workers

3. an excess of 100 workers is only half as bad as a shortfall of 100 workers

No matter what the company does about profit, employment, or pollution, there are two major technological restrictions which *must* be met; these limit the plant's operating level.

So that we can focus our attention on the goal programming aspect of this example, suppose that the following initial formulation has already been completed:

1. the profit in millions of dollars per year is given by $10X_1 + 9X_2 + 14X_3$

2. employment in 100's of employees is given by $4X_1 + 8X_2 + 6X_3$

3. SO_2 emissions in 1000's of Tonnes per annum is $5X_1 + 10X_2 + 7X_3$

4. the plant maximum operating level constraint is

$$4X_1 + 12X_2 + 9X_3 \leq 17$$

5. the plant minimum operating level constraint is

$$2X_1 + 3X_2 + 5X_3 \geq 8$$

7.2.2 Goal Formulation

Since the goal of having a profit of \$45,000,000 is a minimum, overachievement is not of concern. Therefore we only need D_1^-, which we define as the amount of money in millions of dollars by which the profit falls short of the goal. The inequality is:

$$10X_1 + 9X_2 + 14X_3 + D_1^- \geq 45$$

The employment target is 16 hundred workers. Both deviational variables are needed. D_2^- represents the underachievement in hundreds of workers; D_2^+ represents the overachievement in hundreds of workers. The constraint is an equality:

$$4X_1 + 8X_2 + 6X_3 + D_2^- - D_2^+ = 16$$

Finally, there is the goal of reducing the SO_2 emissions to be no more than 20 thousand Tonnes. Underachievement is not of concern, hence we only need D_3^+, which is defined as the amount by which the emissions goal is exceeded, expressed in thousands of Tonnes. The inequality is:

$$5X_1 + 10X_2 + 7X_3 - D_3^+ \leq 20$$

The variables in the objective function are D_1^-, D_2^-, D_2^+, and D_3^+. Now we need their coefficients, We can arbitrarily set a penalty weight of 1 for a \$1,000,000 profit shortfall. To find the other coefficients, we need the statements from the board of directors. We were told that "failing to meet the profit objective by \$1,000,000 is twice as bad as exceeding the SO_2 emissions by 1000 Tonnes". From this it follows that an excess of 1000 Tonnes of SO_2 has a penalty weight of 0.5 (inverse of 2 which comes from "twice as bad"). Then, we see that a shortfall of 125 employees has a penalty weight of 1. Hence, a shortfall of 100 employees has a weight of $\frac{100}{125} \times 1 = 0.8$. Finally, since an excess of 100 workers is only half as bad as an underachievement of 100 workers, the former has a weight of $0.8 \times 0.5 = 0.4$. If desired, these weights can be re-scaled by multiplying each of them by a positive constant. Hence the objective function coefficients are:

Variable	one unit means	Original Weight	Re-scaled Weight
D_1^-	\$1,000,000 under	1	10
D_2^-	100 employees under	0.8	8
D_2^+	100 employees over	0.4	4
D_3^+	1000 T SO_2 over	0.5	5

The entire goal formulation is therefore:

$$
\begin{array}{rrcrl}
\text{minimize} & 10D_1^- + 8D_2^- + 4D_2^+ + 5D_3^+ & & & \\
\text{subject to} & & & & \\
\text{Profit} & 10X_1 + 9X_2 + 14X_3 + D_1^- & \geq & 45 & (1) \\
\text{Employment} & 4X_1 + 8X_2 + 6X_3 + D_2^- - D_2^+ & = & 16 & (2) \\
\text{SO}_2 & 5X_1 + 10X_2 + 7X_3 - D_3^+ & \leq & 20 & (3) \\
\text{Capacity} & 4X_1 + 12X_2 + 9X_3 & \leq & 17 & (4) \\
\text{Min. Production} & 2X_1 + 3X_2 + 5X_3 & \geq & 8 & (5)
\end{array}
$$

all variables must be ≥ 0

7.2.3 Solution

Putting this model onto Excel we have:

	A	B	C	D	E	F	G	H	I	J	K
1		Smelter Model									
2	OFV	X1	X2	X3	D1-	D2-	D2+	D3+			
3	0	0	0	0	10	8	4	5			
4	minimize										
5											
6	Constraints										RHS
7	Profit	10	9	14	1				0	>=	45
8	Employment	4	8	6		1	-1		0	=	16
9	SO2	5	10	7				-1	0	<=	20
10	Capacity	4	12	9					0	<=	17
11	Min. Production	2	3	5					0	>=	8

In cell A3, where we compute the OFV, we can omit the ranges for the X variables, and use just `=SUMPRODUCT(E3:H3,E4:H4)`. However, we need all the variable cells to be included in the computations in column I. In cell I7 we enter `=SUMPRODUCT(B$4:H$4,B7:H7)`, and this is then copied into the range I7:I11. Solving the model by using the Excel Solver we obtain:

	A	B	C	D	E	F	G	H	I	J	K
1		Smelter Model									
2	OFV	X1	X2	X3	D1-	D2-	D2+	D3+			
3	35.25	0	0	0	10	8	4	5			
4	minimize	4.25	0	0	2.5	0	1	1.25			
5											
6	Constraints										RHS
7	Profit	10	9	14	1				45	>=	45
8	Employment	4	8	6		1	-1		16	=	16
9	SO2	5	10	7				-1	20	<=	20
10	Capacity	4	12	9					17	<=	17
11	Min. Production	2	3	5					8.5	>=	8

The solution in brief is:

$$
\begin{aligned}
\mathrm{OFV}^* &= 35.25 \\
X_1^* &= 4.25 \\
D_1^{-*} &= 2.50 \\
D_2^{+*} &= 1.00 \\
D_3^{+*} &= 1.25
\end{aligned}
$$

with all other variables being zero. This solution is interpreted as follows:

1. $X_1^* = 4.25$, hence 4,250,000 Tonnes of ore from concentrator 1 is sent to the smelter. (Nothing is accepted from concentrators 2 or 3.)

2. $D_1^{-*} = 2.50$ means that the profit is \$2,500,000 below its target.

3. Since $D_2^{+*} = 1.00$ the employment is 100 workers above its target.

4. Since $D_3^{+*} = 1.25$ the pollution is 1250 Tonnes above the target.

5. The OFV in this situation is essentially meaningless. It is not measured in dollars, employees, or Tonnes of SO_2; the OFV is only a crude measurement of the board's discomfort with unsatisfied goals, as interpreted by the aforementioned statements.

It would be up to management to decide whether this is a good or bad solution. While no target is met, changing the solution so that one deviation is reduced will result in at least one other deviation being increased. In this sense, the solution is good: the misery has been shared by multiple stakeholders. (The union, however, should be happy, since their membership will increase.)

By playing with the objective function coefficients, management can examine alternate solutions. For example, suppose that we change the coefficient of D_3^+ from 5 to 1000. Re-solving the model we obtain:

	A	B	C	D	E	F	G	H	I	J	K
1		Smelter Model									
2	OFV	X1	X2	X3	D1-	D2-	D2+	D3+			
3	50	0	0	0	10	8	4	1000			
4	minimize	4	0	0	5	0	0	0			
5											
6	Constraints										RHS
7	Profit	10	9	14	1				45	>=	45
8	Employment	4	8	6		1	-1		16	=	16
9	SO2	5	10	7				-1	20	<=	20
10	Capacity	4	12	9					16	<=	17
11	Min. Production	2	3	5					8	>=	8

We see that the pollution goal is met (indeed so is the employment target), but the underachievement of the profit target increases from \$2,500,000 to \$5,000,000. This decrease in profit of \$2,500,000 to obtain a reduction in pollution from 21,250 to 20,000 Tonnes works out to an average cost of \$2,000 per Tonne. Knowing this sort of information is useful when bargaining for the rights for a new pollution abatement technology. Another kind of trade-off which could be calculated is that between pollution and employment. Note that shadow prices do not help us here, because the OFV is not measured in dollars.

Obviously, the difficult part of all this is obtaining the trade-off relationships on which the objective function coefficients are based. A board of directors might contain a major shareholder who is only interested in dividends, and an environmental activist who wants the 20,000 to be not only met, but wishes the target itself to be lowered. In this situation, it might be difficult to come up with a trade-off statement; indeed, even a single individual might have this difficulty.

An alternate procedure is to rank the goals in descending order of importance. This is investigated in the next section.

7.3 Preemptive Goal Programming

In preemptive goal programming with n goals, each goal is ranked from the most important (P_1) to the least important (P_n). Goal i must be satisfied, or should this not be possible, goal i must be as close to being satisfied as possible, before goal $i+1$ can be considered. The technical operations to achieve this are quite straightforward; the difficulty is a managerial one – that of determining not only what the goals should be, but also the ranking of the goals. With n goals to rank, there are $n!$ possible orderings.

7.3.1 Problem Description – Energy Model

An electrical utility generates energy from both hydro (falling water) and thermal sources. The latter include both conventional fossil fuels (coal, oil, and natural gas), and nuclear. The demand for power fluctuates throughout the day and with the seasons, giving rise to a base load with daily and seasonal requirements above this level. By controlling the valves at a hydro-electric station the amount of power can be varied, but thermal plants operate efficiently when producing at a constant rate near the capacity of the plant. Hence thermal plants produce much of the base load, with hydro stations supplying the rest up to the peak demands on the system. It is also possible for a utility to buy or sell energy to a neighbouring utility.

Given its hydro resources and the anticipated demand, the utility needs 12,000 megawatts (MW)[4] to come from thermal sources. Producing more is wasted; producing less means brownouts or purchasing peak power from elsewhere. If cost were the only consideration, they would not use any fossil-fuel plants. It costs about 6.3 cents per kilowatt-hour (kwh) to produce electricity at a fossil-fuel plant but only 5.4 cents per kwh at a nuclear plant. Energy is sold to local distribution companies for 6 cents per kwh, hence the utility loses money on its fossil-fuel thermal plants.

In addition to being more costly, there is more pollution from fossil-fuel plants. Each 1000 MW of fossil-fuel power produces pollution at a rate of 2 units per second, versus 1 unit per second for nuclear. The utility wishes the total pollution rate to be no more than 16 units/second.

[4]The basic unit of energy is the *joule*, which is the energy required to exert a force of one newton over a distance of one metre. Power, which is the *rate* of energy, is measured in *watts*, a watt being a rate of one joule per second. As a practical measure, electrical energy is measured by the kilowatt-hour, a kwh being the equivalent of 3,600,000 joules.

Because of the fear of a Chernobyl type explosion, and the concern about radioactive waste, there is a considerable anti-nuclear movement. Because of this, the public affairs department has recommended that no more than 40% of the thermal power come from nuclear sources.

The utility needs a contribution to profit of about 15 thousand dollars per hour from its thermal plant operations in order to retire its long-term debt.

By building several small plants, any amount of fossil-fuel power can be produced up to 11,000 MW. There is only one approved site for a nuclear power plant, which is limited to 10,000 MW.

7.3.2 Formulation

If all the preceding were hard constraints, we would formulate this problem as a linear model. We begin by defining:

$F =$ amount of fossil-fuel thermal power in 1000's of MW

$N =$ amount of nuclear thermal power in 1000's of MW

For each 1000 MW of fossil-fuel power, the *loss* is

$$1000\text{MW} \times 1000\text{KW/MW} \times (\$.063 - .06)/\text{kwh} = 3 \text{ thousand dollars per hour}$$

For each 1000 MW of nuclear power, the contribution to profit is

$$1000\text{MW} \times 1000\text{KW/MW} \times (\$.06 - .054)/\text{kwh} = 6 \text{ thousand dollars per hour}$$

Contribution to profit is a constraint as well as an objective. Hence the linear model is

$$\begin{array}{rrcrl}
\text{maximize} & -3F + 6N & & & \\
\text{subject to} & & & & \\
\text{Required power} & F + N & = & 12 & (1) \\
\text{Pollution} & 2F + N & \leq & 16 & (2) \\
\text{Proportion} & -0.4F + 0.6N & \leq & 0 & (3) \\
\text{Required profit} & -3F + 6N & \geq & 15 & (4) \\
\text{Fossil-fuel capacity} & F & \leq & 11 & (5) \\
\text{Nuclear capacity} & N & \leq & 10 & (6)
\end{array}$$

$$F, N \geq 0$$

Either by graphing the constraints, or by using the Excel Solver, it is easily verified that there is no feasible solution. (Indeed, this is so even without constraint (4)). This being the case, it is formulated as a goal programming model.

Suppose that the first four constraints can be treated as goal constraints, while the last two are system constraints. Since constraint (1) is now

$$F + N \approx 12$$

We let D_1^- and D_1^+ be the amount in 1000's of megawatts by which the target of 12,000 MW is underachieved or overachieved respectively. Hence

$$F + N + D_1^- - D_1^+ = 12$$

For the next three constraints, we use only one deviational variable:

We let D_2^+ in pollution units per second be the amount by which the pollution level is exceeded, giving

$$2F + N - D_2^+ \leq 16$$

We let D_3^+ be the amount by which the right hand side of the proportion constraint is exceeded:

$$-0.4F + 0.6N - D_3^+ \leq 0$$

Finally, we let D_4^- be the amount in thousands of dollars per hour by which the target for contribution to profit is not met:

$$-3F + 6N + D_4^- \geq 15$$

There are five defined deviational variables and five priorities which need to be established.[5] Consequently, there are $5! = 120$ ways to order the priorities.

It is not necessary to consider each ordering. At the outset, management must decide which is the highest priority, then of the remaining four choose the most important of these, and so on, entailing four $(5-1)$ separate decisions. They might begin, for example, by deciding that above all else their customers will not tolerate the need to ration electrical energy, so the first priority (P_1) is to minimize the underachievement of the 12000 MW, in other words, minimize D_1^-. They might next decide to meet the profit objective, for without that there will be no further development of infrastructure, i.e. minimize D_4^-. Continuing for all five priorities could lead to:

[5] Optionally, there are three more deviational variables which could have been defined (D_2^-, D_3^-, and D_4^+), giving four equality goal constraints, but the number of priorities would remain at five.

Priority	Minimize	Variable
P_1	power shortage	D_1^-
P_2	profit shortage	D_4^-
P_3	excess pollution	D_2^+
P_4	excess in the proportion	D_3^+
P_5	power surplus	D_1^+

In the objective function, each P_i is written followed by the corresponding deviational variable. Doing this, the entire formulation is:

$$\begin{array}{rrcll}
\text{minimize} & P_1(D_1^-)+P_2(D_4^-)+P_3(D_2^+)+P_4(D_3^+)+P_5(D_1^+) & & & \\
\text{subject to} & & & & \\
\text{Required power} & F+N+D_1^- - D_1^+ & = & 12 & (1) \\
\text{Pollution} & 2F+N-D_2^+ & \leq & 16 & (2) \\
\text{Proportion} & -0.4F+0.6N-D_3^+ & \leq & 0 & (3) \\
\text{Required profit} & -3F+6N+D_4^- & \geq & 15 & (4) \\
\text{Fossil-fuel capacity} & F & \leq & 11 & (5) \\
\text{Nuclear capacity} & N & \leq & 10 & (6)
\end{array}$$

$$\text{all variables must be } \geq 0$$

Since the P_i's are not objective function coefficients, the objective function in this context is merely symbolic.[6]

7.3.3 Graphical Solution

Though this model has seven variables in total, it is nevertheless possible to solve this model graphically. This is because it has only two decision variables, the other five being deviational variables. Preemptive goal programming can be solved graphically by solving several sub-problems sequentially. In each sub-problem, the system constraints remain the same; it is only the objective function and the goal constraints which change. At each iteration, the graph is based on the two decision variables. Only one deviational variable is considered at each iteration,

[6]Optionally, the pollution, proportion, and required profit constraints could be written as:

$$\begin{array}{rrcll}
\text{Pollution} & 2F+N+D_2^- - D_2^+ & = & 16 & (2) \\
\text{Proportion} & -0.4F+0.6N+D_3^- - D_3^+ & = & 0 & (3) \\
\text{Required profit} & -3F+6N+D_4^- - D_4^+ & = & 15 & (4)
\end{array}$$

and it occupies a third dimension (which we are trying to minimize to 0) coming out of the main plane.

At the outset, we minimize the deviational variable associated with the first priority, subject to the goal constraint in which this variable appears, and to the system constraints and non-negativity restrictions. In this example, the priority 1 deviational variable is D_1^-, which appears in the first goal constraint (required power). For the rest of the solution we will use constraint numbers only. Variable D_1^+ also appears in constraint (1): we can keep it in if we wish, or remove it to obtain a constraint numbered (1a), which is

$$F + N + D_1^- \geq 12.$$

The latter approach makes the graphical solution more intuitive.

The first sub-problem is therefore:

$$\begin{array}{rrcll} \text{minimize} & D_1^- & & & \\ \text{subject to} & & & & \\ \text{Required power} & F + N + D_1^- & \geq & 12 & (1a) \\ \text{Fossil-fuel capacity} & F & \leq & 11 & (5) \\ \text{Nuclear capacity} & N & \leq & 10 & (6) \\ & & & & \\ & \text{all variables must be } \geq 0 & & & \end{array}$$

This model has three variables, not two. However, since the target value for D_1^- is 0, we can first deal with F and N as if the target value were obtainable. Hence we graph this with F on the horizontal axis and N on the vertical axis. We can think of D_1^- as if it were coming out of the page, with the page itself having $D_1^- = 0$. Hence we plot (1a) in the form

$$F + N \geq 12$$

It is not until we reach the fourth sub-problem that the third dimension will become non-zero.

The feasible region, shown in gold in Figure 7.1, satisfies all constraints in the first sub-problem. Subsequent sub-problems add more constraints, which will gradually shrink the feasible region, until there is none at all in the fourth graph.

In all parts of the feasible region, $D_1^- = 0$, and hence OFV$= 0$. The values of F and N are immaterial (there are multiple optima).

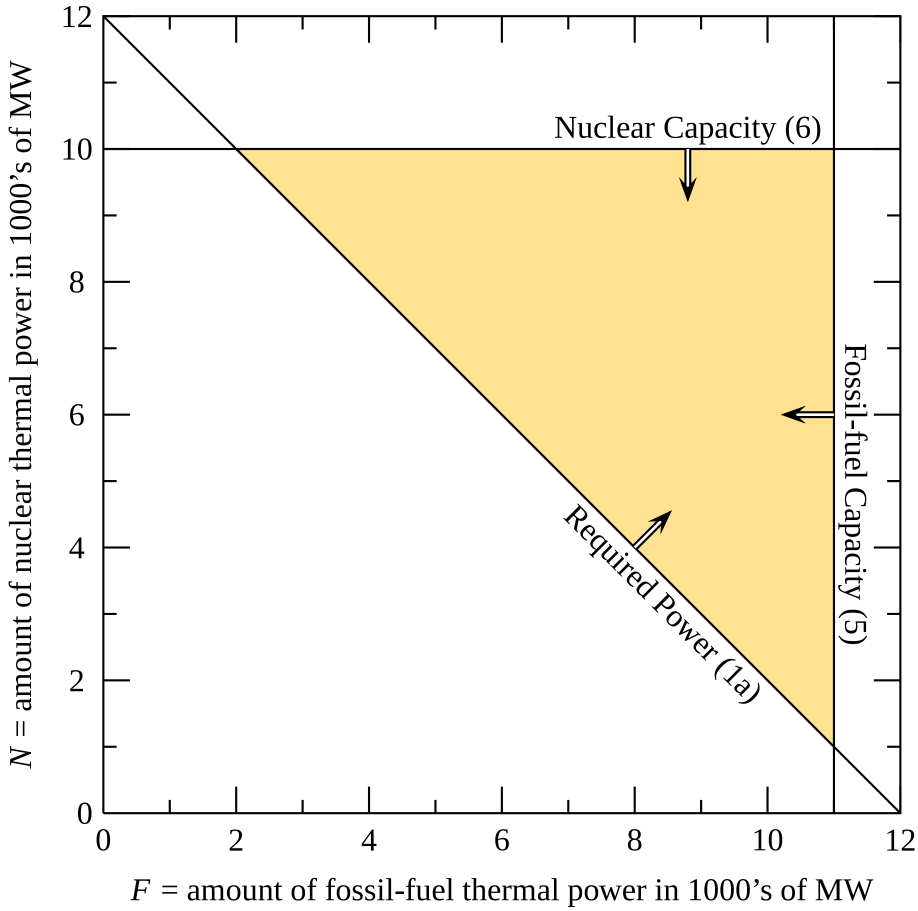


Figure 7.1: Energy Model: 1st Sub-problem

Knowing that it is possible to obtain $D_1^- = 0$, we add $F + N \geq 12$ as a hard constraint, and solve the sub-problem associated with the second priority.

$$
\begin{array}{rrcll}
\text{minimize} & D_4^- & & & \\
\text{subject to} & & & & \\
\text{Required power} & F+N & \geq & 12 & (1a) \\
\text{Required profit} & -3F+6N+D_4^- & \geq & 15 & (4) \\
\text{Fossil-fuel capacity} & F & \leq & 11 & (5) \\
\text{Nuclear capacity} & N & \leq & 10 & (6) \\
 & & & & \\
 & \text{all variables must be } \geq 0 & & &
\end{array}
$$

The graph of the model for the second sub-problem is shown in Figure 7.2. Again, an OFV of 0 has been obtained, D_4^- being 0 everywhere inside the new feasible region. Hence, we add $-3F + 6N \geq 15$ as a hard constraint to the third sub-problem:

$$
\begin{array}{rrcll}
\text{minimize} & D_2^+ & & & \\
\text{subject to} & & & & \\
\text{Required power} & F+N & \geq & 12 & (1a) \\
\text{Pollution} & 2F+N-D_2^+ & \leq & 16 & (2) \\
\text{Required profit} & -3F+6N & \geq & 15 & (4) \\
\text{Fossil-fuel capacity} & F & \leq & 11 & (5) \\
\text{Nuclear capacity} & N & \leq & 10 & (6) \\
 & & & & \\
 & \text{all variables must be } \geq 0 & & &
\end{array}
$$

The graph of the model for the third sub-problem is shown in Figure 7.3. In this third feasible region $D_2^+ = 0$. Hence we add $2F + N \leq 16$ as a hard constraint to the fourth sub-problem:

$$
\begin{array}{rrcll}
\text{minimize} & D_3^+ & & & \\
\text{subject to} & & & & \\
\text{Required power} & F+N & \geq & 12 & (1a) \\
\text{Pollution} & 2F+N & \leq & 16 & (2) \\
\text{Proportion} & -0.4F+0.6N-D_3^+ & \leq & 0 & (3) \\
\text{Required profit} & -3F+6N & \geq & 15 & (4) \\
\text{Fossil-fuel capacity} & F & \leq & 11 & (5) \\
\text{Nuclear capacity} & N & \leq & 10 & (6) \\
 & & & & \\
 & \text{all variables must be } \geq 0 & & &
\end{array}
$$

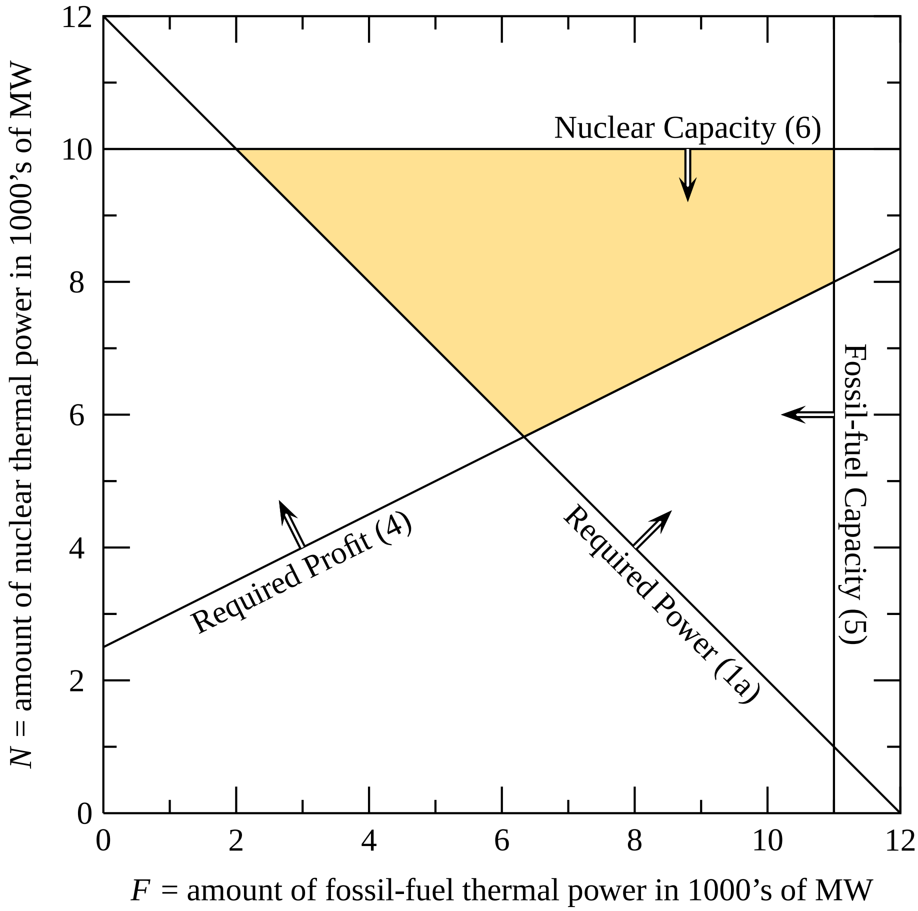


Figure 7.2: Energy Model: 2nd Sub-problem

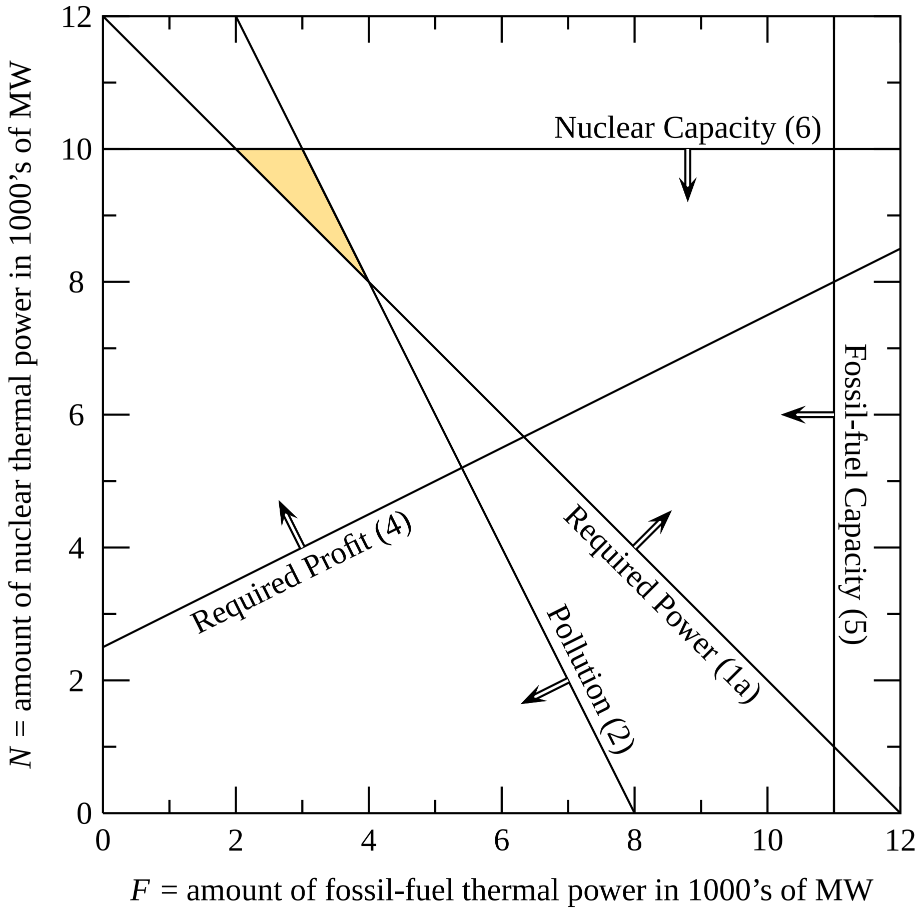


Figure 7.3: Energy Model: 3rd Sub-problem

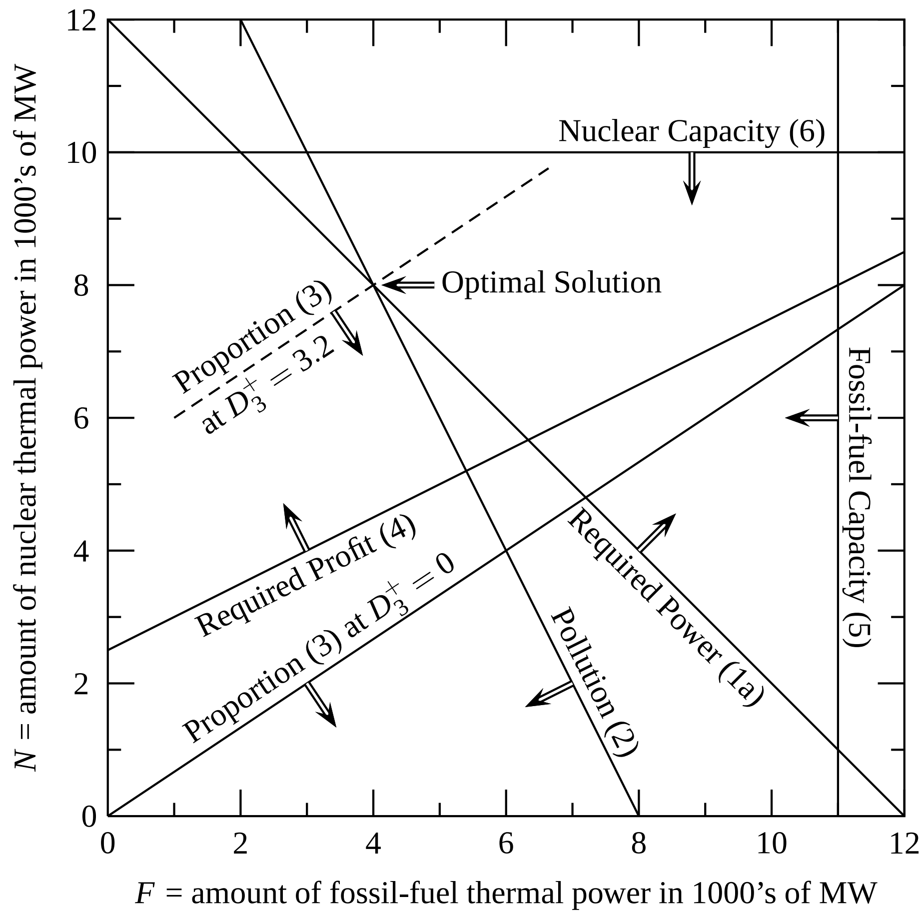


Figure 7.4: Energy Model: 4th Sub-problem

The graph of the model for the fourth sub-problem is shown in Figure 7.4. The boundary of constraint (3) with $D_3^+ = 0$ which passes through the origin is seen to be infeasible. Hence another constraint (3) was drawn, parallel with the first, so that it just touches the feasible region of the third sub-problem. (This operation is like the creation of parallel isovalue lines in the graphical technique for linear models.) Doing this minimizes the value of D_3^+. This results in a unique feasible solution with respect to F and N,[7] which occurs at the boundaries of constraints (1) and (2).

[7]Over all three variables F, N, and D_3^+, the feasible region is of infinite size, but because we minimize with respect to D_3^+, we obtain a unique solution for F and N.

By linear algebra, we determine that $F = 4$, and $N = 8$. By substituting into (3), we see that D_3^+ is minimized at 3.2. However, since (3) is a *proportion* constraint, the 3.2 is not of great practical significance. Instead, we are interested in

$$\begin{aligned} \frac{N}{F+N} &= \frac{8}{4+8} \\ &= 66.7\% \end{aligned}$$

meaning that the target of 40% has been exceeded by 26.7%.

Given that $F = 4$ and $N = 8$, there is no point in graphing sub-problem (5). Since the minimum value of D_3^+ was found to be 3.2, this value is added to both sides of the proportion constraint to obtain:

$$-0.4F + 0.6N \leq 3.2$$

Incorporating this constraint, the next subproblem is:

$$\begin{array}{rrcll} \text{minimize} & D_1^+ & & & \\ \text{subject to} & & & & \\ \text{Required power} & F+N & \geq & 12 & (1a) \\ \text{Required power} & F+N-D_1^+ & \leq & 12 & (1b) \\ \text{Pollution} & 2F+N & \leq & 16 & (2) \\ \text{Proportion} & -0.4F+0.6N & \leq & 3.2 & (3) \\ \text{Required profit} & -3F+6N & \geq & 15 & (4) \\ \text{Fossil-fuel capacity} & F & \leq & 11 & (5) \\ \text{Nuclear capacity} & N & \leq & 10 & (6) \end{array}$$

$$\text{all variables must be } \geq 0$$

Either by plugging in the values of F and N into (1b), or by noting directly that constraint (1b) is binding, we see that $D_1^+ = 0$. Hence, although the fourth priority goal is not met, the fifth priority goal is met.

To solve this problem, all we needed to do is one make graph, i.e. the fourth one of the four shown here. However, to illustrate the process sequentially, four graphs were made. On the first three graphs, the existence of the region highlighted in gold means that all constraints up to that point can be satisfied. When we reached the fourth graph, there is no solution any solution in the (F,N) plane, and one of the deviational variables became non-zero.

7.3.4 Solution Using the Excel Solver

Instead of solving each sub-problem by the graphical method, we could use a computer. It is possible to solve the entire problem with one optimization, but this requires a modified simplex algorithm.[8] Most optimization packages which are based on algebraic modeling contain this modified algorithm, but the Excel Solver is not set up for this.

However, the ordinary simplex algorithm within Excel Solver can still be used, with a bit of extra work. As with the graphical method, five sub-problems are solved for this example, and hence up to five optimizations may be required. For each optimization, the user alters the objective function and the constraints, just as with the graphical method. For each optimization model, the value of that model's optimized deviational variable (be it 0 or more than 0) is frozen for the subsequent optimization models.

We put the algebraic model into Excel as follows:

	A	B	C	D	E	F	G	H	I	J	K	L
1	OFV			Energy Model								
2	minimize			P1	P2	P3	P4	P5			Model	
3	0	F	N	D1-	D4-	D2+	D3+	D1+			1	
4												
5	Constraints										RHS	
6	Required power	1	1	1				-1	0	=	12	1
7	Pollution	2	1			-1			0	<=	16	2
8	Proportion	-0.4	0.6				-1		0	<=	0	3
9	Required profit	-3	6		1				0	>=	15	4
10	Fossil-fuel capacity	1							0	<=	11	5
11	Nuclear capacity		1						0	<=	10	6

There are some things which are a bit different from what one might expect.

1. The deviational variables are ordered according to the priorities, rather than the order in which they appear in the constraints.

2. There is no row of objective function coefficients, because each deviational variable is considered sequentially.

[8] The technical details of this modified algorithm can be found, for example, in Wayne L. Winston, *Operations Research: Applications and Algorithms*. (4th edition, Cengage, 2004).

3. In the range of variable cells (B4:H4), one of the cells for the deviational variables is to be minimized in each model. At the outset when we are dealing with Priority 1, this is the value of D_1^- in cell D4. Hence the OFV in cell A3 is computed simply as `=D4`.

4. We need to run multiple models, so for the sake of clarity the model number is indicated in cell K3.

5. Only some of the constraints will appear in each model. At the outset, these are constraints (1), (5), and (6). These are identified by the green cells in column L.

By contrast, filling in Column I is done in the usual way. We put `=SUMPRODUCT(B$4:H$4,B6:H6)` into cell I6, and then copy this into the range I6:I11. Also, most of what we do in the Solver is what we've done all along. The exceptions are: (1) the third step on the following list, in which some of the yellow cells are not declared as "changing cells"; and (2) the fourth step, in which only some of the constraints are entered. Note that unlike the graphical solution, we have kept the first constraint as it was in the algebraic model. However, an alternate approach which does follow the graphical solution appears on page 312.

1. Make the "Set objective" cell `$A$3`.

2. Click the "min" radio button.

3. The changing cells are `$B$4:$D$4,$H$4`. Note that we have excluded the cells for the values of D_4^-, D_2^+, and D_3^+.

4. Subject to the constraints `I6 = K6` and `I10:I11` $\leq$ `K10:K11`. (These are the constraints identified in green on the extreme right.)

5. Click on the box next to "Make Unconstrained Variables Non-Negative".

6. Select the Simplex LP and the click on the "Solve" button.

We obtain:

	A	B	C	D	E	F	G	H	I	J	K	L
1	OFV			Energy Model								
2	minimize			P1	P2	P3	P4	P5			Model	
3	0	F	N	D1-	D4-	D2+	D3+	D1+			1	
4		11	1	0				0				
5	Constraints										RHS	
6	Required power	1	1	1				-1	12	=	12	1
7	Pollution	2	1			-1			23	<=	16	2
8	Proportion	-0.4	0.6				-1		-3.8	<=	0	3
9	Required profit	-3	6		1				-27	>=	15	4
10	Fossil-fuel capacity	1							11	<=	11	5
11	Nuclear capacity		1						1	<=	10	6

To make the second model we:

1. Change cell K3 from 1 to 2.
2. Change cell A3 from `=D4` to `=E4`.
3. The second-priority deviational variable (which is D_4^- in column E) appears in row 9, so make cell L9 green.
4. In the Solver, we need to delete cell D4 from the list of changing cells, and add cell E4 to this list. It is important to note the number in cell D4, which is 0 from the first optimization, must remain at 0. The list is now `$B$4:$C$4,$E$4,$H$4`.
5. In the Solver, add the relationship of row 9 to the list of constraints: I9 $\geq$ K9.

We solve the model to obtain:

	A	B	C	D	E	F	G	H	I	J	K	L
1	OFV			Energy Model								
2	minimize			P1	P2	P3	P4	P5			Model	
3	0	F	N	D1-	D4-	D2+	D3+	D1+			2	
4		6.33	5.67	0	0			0				
5	Constraints										RHS	
6	Required power	1	1	1				-1	12	=	12	1
7	Pollution	2	1			-1			18.3	<=	16	2
8	Proportion	-0.4	0.6				-1		0.87	<=	0	3
9	Required profit	-3	6		1				15	>=	15	4
10	Fossil-fuel capacity	1							6.33	<=	11	5
11	Nuclear capacity		1						5.67	<=	10	6

To make the third model we:

1. Change cell K3 from 2 to 3.

2. Change cell A3 from `=E4` to `=F4`.

3. The third-priority deviational variable (which is D_2^+ in column F) appears in row 7, so make cell L7 green.

4. In the Solver, replace E4 with F4. The 0 in cell E4 remains at this value. The list is now `$B$4:$C$4,$F$4,$H$4`.

5. In the Solver, add the relationship of row 7 to the list of constraints: I7 $\leq$ K7.

We solve the model to obtain:

	A	B	C	D	E	F	G	H	I	J	K	L
1	OFV			Energy Model								
2	minimize			P1	P2	P3	P4	P5			Model	
3	0	F	N	D1-	D4-	D2+	D3+	D1+			3	
4		4	8	0	0	0		0				
5	Constraints										RHS	
6	Required power	1	1	1				-1	12	=	12	1
7	Pollution	2	1			-1			16	<=	16	2
8	Proportion	-0.4	0.6				-1		3.2	<=	0	3
9	Required profit	-3	6		1				36	>=	15	4
10	Fossil-fuel capacity	1							4	<=	11	5
11	Nuclear capacity		1						8	<=	10	6

To make the fourth model we:

1. Change cell K3 from 3 to 4.

2. Change cell A3 from `=F4` to `=G4`.

3. The fourth-priority deviational variable (which is D_3^+ in column G) appears in row 8, so make cell L8 green.

4. In the Solver, replace F4 with G4. The 0 in cell F4 remains at this value. The list is now `$B$4:$C$4,$G$4,$H$4`.

5. In the Solver, add the relationship of row 8 to the list of constraints: I8 $\leq$ K8.

We solve the model to obtain:

	A	B	C	D	E	F	G	H	I	J	K	L
1	OFV			Energy Model								
2	minimize			P1	P2	P3	P4	P5			Model	
3	3.2	F	N	D1-	D4-	D2+	D3+	D1+			4	
4		4	8	0	0	0	3.2	0				
5	Constraints										RHS	
6	Required power	1	1	1				-1	12	=	12	1
7	Pollution	2	1			-1			16	<=	16	2
8	Proportion	-0.4	0.6				-1		-0	<=	0	3
9	Required profit	-3	6		1				36	>=	15	4
10	Fossil-fuel capacity	1							4	<=	11	5
11	Nuclear capacity		1						8	<=	10	6

We have obtained $D_3^+ = 3.2$, with $F = 4$, and $N = 8$. It may not be obvious, but we do not need to run the fifth model; if we do, we will obtain $D_1^+ = 0$.

An Alternate Solution Approach This alternate approach follows the approach of the graphical solution. At the outset, we replace constraint (1) with constraint (1a), in which the D_1^+ variable does not appear, and the = sign is replaced by $\geq$. We put >= in cell J6, and `1a` in cell L6. When we perform the first optimization, we do not allow the Solver the change the cell for the value of D_1^+ (which is cell H4), and now the first constraint is $\geq$. We set cell A3 to be `=D4`.

1. Make the "Set objective" cell `$A$3`.

2. Click the "min" radio button.

3. The changing cells are `$B$4:$D$4`. Note that we have excluded the cells for the values of D_4^-, D_2^+, D_3^+, and D_1^+.

4. Subject to the constraints `I6` $\geq$ `K6` and `I10:I11` $\leq$ `K10:K11`. (These are the constraints identified in green on the extreme right.)

5. Click on the box next to "Make Unconstrained Variables Non-Negative".

6. Select the Simplex LP and the click on the "Solve" button.

We obtain:

	A	B	C	D	E	F	G	H	I	J	K	L
1	OFV			Energy Model								
2	minimize			P1	P2	P3	P4	P5			Model	
3	0	F	N	D1-	D4-	D2+	D3+	D1+			1	
4		11	1	0								
5	Constraints										RHS	
6	Required power	1	1	1				-1	12	>=	12	1a
7	Pollution	2	1			-1			23	<=	16	2
8	Proportion	-0.4	0.6				-1		-3.8	<=	0	3
9	Required profit	-3	6		1				-27	>=	15	4
10	Fossil-fuel capacity	1							11	<=	11	5
11	Nuclear capacity		1						1	<=	10	6

We see that cell H4 is blank, rather than 0. Other than this, there is no difference from what we did before. In each of the second, third, and fourth optimizations, cell H4 is excluded from the list of changing cells, and constraint (1a) remains $\geq$. Here is a summary for the first four optimizations:

K3	Set A3	Changing Cells	Constraints
1	`=D4`	`$B$4:$D$4`	`I6` $\geq$ `K6` and `I10:I11` $\leq$ `K10:K11`
2	`=E4`	`$B$4:$C$4,$E$4`	as for K3 = 1 plus `I9` $\geq$ `K9`
3	`=F4`	`$B$4:$C$4,$F$4`	as for K3 = 2 plus `I7` $\leq$ `K7`
4	`=G4`	`$B$4:$C$4,$G$4`	as for K3 = 3 plus `I8` $\leq$ `K8`

The optimized models for K3 = 2, 3, 4 will be the same as shown earlier, except for cell H4 being blank rather then 0.

To begin the fifth optimization, we change constraint (1a) back to its original form as (1), an equality constraint. Now we minimize cell A3 which has been set equal to H4, allowing cells B4:C4 and H4 to vary; we obtain a 0 in H4 (and hence a 0 in A3). This is exactly what we obtained earlier in the fourth optimization. The fifth optimization of this alternate approach is:

	A	B	C	D	E	F	G	H	I	J	K	L
1	OFV			Energy Model								
2	minimize			P1	P2	P3	P4	P5			Model	
3	0	F	N	D1-	D4-	D2+	D3+	D1+			5	
4		4	8	0	0	0	3.2	0				
5	Constraints										RHS	
6	Required power	1	1	1				-1	12	=	12	1
7	Pollution	2	1			-1			16	<=	16	2
8	Proportion	-0.4	0.6				-1		-0	<=	0	3
9	Required profit	-3	6		1				36	>=	15	4
10	Fossil-fuel capacity	1							4	<=	11	5
11	Nuclear capacity		1						8	<=	10	6

7.4 The Economic Order Quantity (EOQ) Model

Up to this point all the models that we have made have had linear objective functions and constraints. By this, we mean that every expression is of the form "coefficient $\times$ variable + coefficient $\times$ variable". Sometimes, however, we cannot make a model in this simple form. Instead, we have things such as variables which are squared, or variables which are multiplied by other variables. When this happens, we have a nonlinear model.

In this section we will make simple inventory model. We shall see that to solve this model we need to optimize a nonlinear function.

7.4.1 Background Information

Companies often have several types of inventory: supplies and/or raw materials, work-in-progress, and finished goods. There are both benefits and costs to having inventories.

One of the benefits is that an inventory helps deal with uncertainty. For example, a car dealer will have many cars on the lot, because if many customers come in at once, the dealer wants to be able to sell a car immediately to each one. It takes several weeks to order a car from the factory, and some customers would go to another dealer rather than wait.

A related benefit of inventory is that it helps deal with fluctuations in the demand, even when such fluctuations are known in advance. A rental car agency often has many cars available on the weekend, simply because the demand is lower on the weekends. The fluctuation in demand creates an inventory of cars on Saturday and Sunday.

Another reason to have inventory is that it is the result of obtaining a quantity discount. We need a litre of oil and see it priced at \$1.30. However, a case containing 24 litres is just \$20.40 (85 cents per litre). If we buy the case, we may end up with several years' supply of oil.

Even when there is no quantity discount from the supplier, companies often order in bulk because it spreads the overhead cost of an order over a large quantity. This cost, often called the "ordering cost", consists of clerical time, paperwork, and the cost of the time to obtain the signatures on the order form.

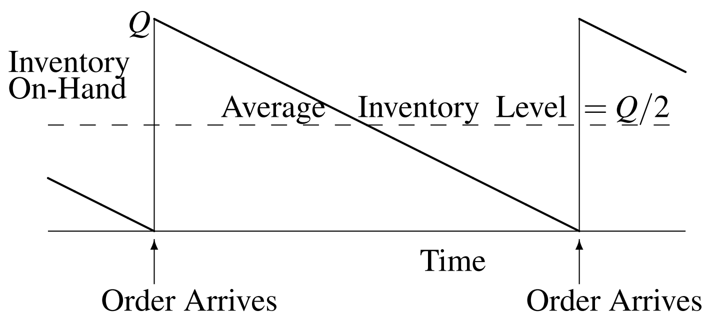
The cost of keeping inventory takes many forms, and depends highly on the commodity. One obvious cost is that of storage. There is the cost of the warehouse, security, insurance and so on. Another major cost is that of tied-up capital. A million dollars worth of cars represents foregone interest that could be earned on the money; the cost is even greater if the money for the inventory of cars has to be borrowed. For some products there is a cost of spoilage (e.g. food, drugs), while for other products there is cost associated with obsolescence (e.g. computers, software).

In the following problem description we consider a very elementary inventory model.

7.4.2 Description

In the most elementary model, there is no uncertainty. Furthermore, demand is assumed to be constant over time. The "lead time", which is the time from placing an order to the time at which it is delivered, is known and constant, and when the order arrives, it arrives all at once. No shortages are allowed, and no quantity discounts are available.

Because of these factors, a repetitive pattern emerges in which the order size, denoted as Q, never varies. The order arrives just as the inventory has become depleted, producing a "sawtooth" pattern as shown in the following diagram.



The order quantity Q is the only unknown of the problem. In developing this model we will not use specific numbers for the coefficients, but instead will use ***parameters***, enabling us to find a general formula for the problem. A parameter can change from problem to problem, but for a particular problem a parameter is a known constant; they are distinguished from variables by putting them in small letters. In this model there are four parameters, whose symbols are::

d	annual demand
c_o	the cost of placing an order
c_h	the cost of holding one unit in inventory for a year
c_p	the cost to purchase one unit

Given all the above, what value of Q minimizes the total cost?

7.4.3 Formulation

We begin by finding an inventory cost function $f(Q)$, defined for $Q > 0$, which has three components.
(1) First, there is the ordering cost. The number of orders per year is $\frac{d}{Q}$. Since the ordering cost of each order is c_o, the annual ordering cost is

$$\text{ordering cost} = c_o \frac{d}{Q}$$

(2) Secondly, there is the holding cost. As shown on the diagram, the average number of units in inventory over the year is $\frac{Q}{2}$. Since the cost to hold one unit in inventory for a year is c_h, the annual holding cost is

$$\text{holding cost} = c_h \frac{Q}{2}$$

(3) Thirdly, there is the cost of purchasing. Each unit costs c_p, and d units are ordered in total, so the annual purchasing cost is simply

$$\text{purchase cost} = c_p d$$

The total inventory cost function is therefore:

$$f(Q) = c_o \frac{d}{Q} + c_h \frac{Q}{2} + c_p d \tag{7.1}$$

It is shown later using differential calculus that this function is minimized at:

$$Q = \sqrt{\frac{2c_o d}{c_h}}$$

This formula is called the ***economic order quantity*** formula, or ***EOQ*** formula for short. It appears in many places in business textbooks. As important as this formula is, one needs to remember the idealistic assumptions on which it is based.

7.4.4 Numerical Example

A company uses 100,000 boxes of paper each year. It costs \$50 to place an order. To hold one box of paper in inventory for a year costs \$2.50. A box of paper costs \$35. How big should each order be, and what is the total annual cost?

The parameters of the model are $d = 100,000$, $c_o = \$50$, $c_h = \$2.50$, and $c_p = \$35$. We use the EOQ formula as follows:

$$\begin{aligned} Q &= \sqrt{\frac{2c_o d}{c_h}} \\ &= \sqrt{\frac{2(50)100,000}{2.50}} \\ &= \sqrt{4,000,000} \\ &= 2000 \end{aligned}$$

Each time an order is placed, it should be for 2000 boxes. The number of orders per year is

$$\begin{aligned} \frac{d}{Q} &= \frac{100,000}{2000} \\ &= 50 \end{aligned}$$

Note that c_p is irrelevant as far as determining Q is concerned. However, it is necessary for determining the value of $f(Q)$.

$$\begin{aligned} f(Q) &= c_o \frac{d}{Q} + c_h \frac{Q}{2} + c_p d \\ &= 50\left(\frac{100,000}{2000}\right) + 2.50\left(\frac{2000}{2}\right) + 35(100,000) \\ &= \$2500 + \$2500 + \$3,500,000 \\ &= \$3,505,000 \end{aligned}$$

Note that both the optimal ordering cost (which is \$2500) and the optimal holding cost (which is \$2500) are the same. This is not a coincidence – this property that these two costs are equal at the point of optimality is always true, and can help act as a check on the numerical calculations.

7.5 Nonlinear Optimization: Introduction

In this section we examine models in which the objective function may be a nonlinear function of many variables, and we seek to either maximize or minimize $f(X_1, X_2, \ldots, X_n)$. This will often be subject to a set of linear or nonlinear constraints. To accomplish anything with this subject we will need to use the Solver in Excel to solve the models.

The rest of this section is organized as follows.

1. For optional use, we present a quick review of single-variable differential calculus. This includes the derivation of the EOQ (economic order quantity) formula.

2. The use of the Excel Solver for nonlinear functions is introduced.

3. Several single-variable problems are modeled, and all are solved using the Excel Solver.

4. Some examples of problems with multiple variables are discussed.

7.5.1 Traditional Optimization (Optional)

Overview

The reader will have presumably completed a course in differential calculus, in which an unconstrained function of a single variable is optimized. The process is:

1. Model the problem using a single variable X, to create a function $f(X)$ that we seek to optimize (i.e., maximize or minimize depending on the situation).

2. Using the rules of differentiation, find the first derivative $f'(X)$. Some basic rules are that the derivative of $f(X) = aX^n$ is $f'(X) = naX^{n-1}$, and the derivation of $f(X) = u(X) + v(X)$ is $f'(X) = u(X) + v(X)$. Many other rules are given on page 516.

3. Set $f'(X) = 0$, and solve this to obtain solution $\overline{X}$.

4. Find the second derivative $f''(X)$.

5. Evaluate $f''(X)$ at $X = \overline{X}$. If $f''(\overline{X}) > 0$, then the function has a local minimum at $X = \overline{X}$. If $f''(\overline{X}) < 0$, then the function has a local maximum at $X = \overline{X}$. If $f''(\overline{X}) = 0$, then further testing is required to determine whether this point is a local maximum, a local minimum, or neither of these. The rules for further testing are presented on page 518.

We now use this procedure to solve for the EOQ formula.

Minimizing $f(Q)$ to obtain the EOQ Formula

Earlier we considered a simple inventory model with the following total inventory cost function:

$$f(Q) = c_o \frac{d}{Q} + c_h \frac{Q}{2} + c_p d \tag{7.2}$$

Here we show that this function is minimized at:

$$Q = \sqrt{\frac{2c_o d}{c_h}}$$

Solution By solving this problem analytically, we obtain the solution for any values of the parameters. This is an immensely useful result.

To find the value of Q which minimizes $f(Q)$ we obtain the first derivative.

$$f'(Q) = -\frac{c_o d}{Q^2} + \frac{c_h}{2} + 0$$

At $f'(Q) = 0$,

$$\begin{aligned}
-\frac{c_o d}{Q^2} + \frac{c_h}{2} &= 0 \\
\frac{c_h}{2} &= \frac{c_o d}{Q^2} \\
Q^2 &= \frac{2c_o d}{c_h} \\
Q &= \sqrt{\frac{2c_o d}{c_h}}
\end{aligned}$$

The second derivative of $f(Q)$, which is the first derivative of $f'(Q)$, is

$$\begin{aligned}
f''(Q) &= -(-2)c_0 d Q^{-3} + 0 \\
&= \frac{2c_o d}{Q^3} \\
&> 0 \qquad\qquad \text{for all } Q > 0
\end{aligned}$$

Hence the function $f(Q)$ is minimized at

$$Q = \sqrt{\frac{2c_o d}{c_h}} \tag{7.3}$$

Limitations of the Analytical Approach

Analytically-based solution methods are limited as follows:

1. It requires a course in Calculus just to learn how to solve single-variable problems.

2. Sometimes, even single-variable problems are not solvable in a closed-form expression. For example, suppose that we wish to minimize, for $X > 0$, the following function:
$$f(X) = \frac{10}{X} + \frac{e^{X/3}}{2}$$
We find the first derivative to be:
$$f'(X) = -\frac{10}{X^2} + \frac{e^{X/3}}{6}$$
Seeking the stationary point of the function, we then set this equal to 0.
$$-\frac{10}{X^2} + \frac{e^{X/3}}{6} = 0$$
We are unable to obtain a closed-form expression for X.

3. Even when a single-variable example is solvable, to find the solution might be long and tedious.

4. Learning how to analytically optimize a function of more than one variable requires a second course in differential calculus, and solutions may be difficult to obtain.

5. Adding constraints adds yet another level of complication.

While the parameter-based EOQ formula derived above had to be solved analytically, whenever we have a numerical example it can be solved using the GRG algorithm which is built into the Excel Solver.

7.5.2 Using the Excel Solver

Suppose that we wish to minimize, for $X > 0$, the following function:

$$f(X) = \frac{10}{X} + \frac{e^{X/3}}{2}$$

There is no closed-form expression that can be obtained using analytical calculus, so it needs to be solved numerically. We can solve the problem using the GRG (Generalized Reduced Gradient) nonlinear algorithm which is built into the Excel Solver.

To do this we reserve a cell in Excel in which the Solver will write the computed value of X. In another cell, we write the formula above, replacing X with its cell reference. For example, we could use cell A1 for the value of X. Here we use the Excel function **EXP** for finding the numerical value of e^X. Then say in cell B1 we would write the formula in Excel's syntax, which is:

$$= 10/\mathrm{A1} + \mathrm{EXP}(\mathrm{A1}/3)/2$$

With nothing in cell A1, this will return an error message because the default value of 0 in A1 causes a division by 0 problem. Typing any positive number in A1 will eliminate this problem. Going to the Solver we need to set the solving method to the "GRG Nonlinear" algorithm, rather than the simplex algorithm that we have been using up till now. We ask the Solver to minimize objective cell B1, with cell A1 being the variable cell. Doing this we obtain 3.986121 in cell A1, and 4.396783 in cell B1. Hence $f(X)$ is minimized at $X = 3.986121$, with $f(X) = 4.396783$.

This is just the beginning of what using the GRG algorithm in the Solver can accomplish. We can solve problems with many variables, and we can add constraints too.

7.5.3 Multiple Variables and Constraints

The conditions for local optimality when there are multiple variables and constraints are very complex, and they will not be given here.[9]

While local optimality is necessary, it is not sufficient. For global optimality, we need to be minimizing a convex function (or maximizing a concave function) over a convex feasible region. A function is convex (concave) if the line segment between any two points on the function lies entirely above (below) the function. A region is convex if we can take any two points in the region, draw a line between them, and all points on the line between the two points are also in the region. For example, a sphere is a convex region. A doughnut, however, is not convex. A very important special case happens when all the constraints are linear. Assuming that a feasible region exists, it will be a convex region.

[9]These conditions were discovered by Kuhn and Tucker and published in 1951, but it was later discovered that they were originally discovered (but not published in the open literature) by Karush in 1939. These conditions are now known as the Karush-Kuhn-Tucker (or KKT) conditions. They are described at https://en.wikipedia.org/wiki/Karush-Kuhn-Tucker_conditions.

When the Excel Solver is used, it solves to find a local point of optimality, and verifies that the conditions for local optimality are satisfied at that point. However, the Solver has no way of telling if the feasible region is convex, nor can it tell if the function being optimized is convex (for minimization) or concave (for maximization). Unless the user has knowledge about these things, the solution found by the Solver cannot be guaranteed to be correct, except in the sense that it's better than all neighbouring points.

7.6 Single-Variable Applications

In this section we examine some business applications of single variable differential calculus. Each application begins with a description of a situation, and from this we must obtain a function that needs to be maximized or minimized. The numerical solution is then obtained by using the GRG algorithm in the Excel Solver.

7.6.1 Price Determination

Description

At a nominal cost of \$1, an entrepreneur purchased an historic lighthouse that was to be demolished. After paying \$20,000 for renovations, he opened it to the public, charging \$2 per person. Attendance has averaged about 600 visitors per week. A survey was taken of visitors to the area, some of whom visited the lighthouse, but others who did not. The survey suggests that 200 customers would be lost each week for each \$1 per person increase in the price, and that 200 customers would be gained each week for each \$1 decrease in the price. It can be assumed that the relationship between demand and price is linear, and that the cost to operate the lighthouse is independent of the number of visitors. What price per person maximizes the weekly revenue (and hence the operating profit)?

Formulation

The last sentence of the problem description suggests that we need to find a revenue or profit function whose argument is the price per person. Hence,

Let P be the price charged per person.

We also need to know the weekly demand, which depends on the price. Hence, Let D be the weekly demand.

We now have two unknowns. We will write D in terms of P, and thereby reduce the problem to one unknown. To find D in terms of P, think of D as being its current value (which is 600) plus/minus a correction term for when P is not at its current value (which is 2). This can be written either as

$$D = 600 + 200(2 - P)$$

or as

$$D = 600 - 200(P - 2)$$

Whichever form we use, it simplifies to

$$D = 1000 - 200P$$

The revenue is the demand multiplied by the price, or $D \times P$. The \$20,000 spent on the property is a sunk cost which is irrelevant to the question at hand. Substituting for D, the product $D \times P$ becomes a function of P alone:

$$f(P) = (1000 - 200P)P$$

Solution using the Excel Solver

If we use cell A1 for the value of P, we would enter `=(1000-200*A1)*A1` into say cell B1, and then ask the Solver to maximize B1 by varying cell A1, with the GRG algorithm being invoked. The solution is $P = 2.5$, with $f(P) = 1250$. At this price per person of \$2.50 the number of people who visit the lighthouse will be

$$\begin{aligned} D &= 1000 - 200(2.5) \\ &= 1000 - 500 \\ &= 500 \end{aligned}$$

From the Solver, the revenue is \$1,250. We can verify this by multiplying D and P:

$$\begin{aligned} D \times P &= 500 \times \$2.50 \\ &= \$1250 \end{aligned}$$

The optimal solution is to charge \$2.50 per person, thereby attracting 500 visitors per week, for a weekly revenue of \$1250. [This compares with a status quo weekly revenue of $600 \times \$2 = \$1200$.]

7.6.2 The Optimal Speed of a Truck

Description

Excluding the cost of fuel, it costs \$34 per hour to operate a truck (labour, tied-up capital). The gasoline consumption (measured in litres per 100 kilometres) depends upon the speed of the truck. We let X represent the speed of the truck in km/hour. Where $X \geq 60$ (when the truck is in its top gear), the fuel consumption has been measured to be 43 L/100 km at 60 km/h, increasing by 0.3 L/100 km for each 1 km/h increase in the speed above 60 km/h. Gasoline costs \$1.40 per litre. The speed limit is 100 km/hour. What speed minimizes the total cost (per given distance) of operating the truck?

Formulation

The easiest way to proceed is to consider the cost of a 100 km trip. The fuel cost is the price per litre (which is \$1.40) multiplied by the number of litres consumed (a function of the speed X). The fuel consumption (in L/100 km) is $43+0.3(X-60)$. Hence the fuel cost is $1.4(43+0.3(X-60))$. The non-fuel cost is \$34 per hour. At a speed of X km/hour, the time required to drive 100 km is $100/X$ hours. Hence the non-fuel cost while driving the 100 km is \$34 per hour multiplied by $100/X$ hours, which is $34(100)/X$. Hence the total cost for a 100 km trip is

$$f(X) = 1.4(43+0.3(X-60)) + 34(100)/X$$

This expression could be simplified to $f(X) = 35+0.42X+3400/X$. However, the first expression preserves the original data, so if for example the price per litre changes, we easily see how to modify the expression.

Implicit in this relationship is that the speed of the truck cannot be negative ($X \geq 0$). Moreover, the truck is assumed to be in top gear ($X \geq 60$). This makes the requirement that X be ≥ 0 redundant, though the usual practice is to state it explicitly nevertheless. Also, we will assume that the speed limit will be obeyed ($X \leq 100$). Therefore, this is a case of constrained optimization:

$$\begin{aligned}
\text{minimize } f(X) &= 1.4(43+0.3(X-60)) + 34(100)/X \\
\text{subject to} & \\
X &\geq 0 \\
X &\geq 60 \\
X &\leq 100
\end{aligned}$$

Solution using the Excel Solver

Letting the value of X be in cell A1, we could enter the cost into say cell B1, and we could put the 60 and the 100 into cells C1 and D1. In cell B1 the formula would be `=1.4*(43+0.3*(A1-60))+34*100/A1`. In the Solver we would minimize cell B1 by varying cell A1, subject to the constraints A1 $\geq$ C1 and A1 $\leq$ D1. The theoretical solution is $X = 89.9735$ with $f(X) = 110.578$. The truck should therefore be driven at about 90 km/hour.

7.6.3 Optimal Level of Production

Description

The selling price of an item is \$6 per unit. The units are made in a factory which has three types of costs: a daily fixed cost of \$8000; a variable cost of \$2 per unit; and a congestion cost which depends on the level of production. The plant has a physical capacity of 9,999 units per day; at 10,000 units per day the cost is infinite because the capacity has been exceeded. Defining X to be the number of units produced each day, someone has gone into the plant to determine the daily congestion cost, which is:

$$\frac{X}{10{,}000 - X} \qquad 0 \leq X \leq 9999$$

We wish to determine the daily production level which maximizes the profit.

Formulation

The daily profit is the daily revenue minus the daily cost. From the information in the problem description we obtain:

$$f(X) = 6X - \left(8000 + 2X + \frac{X}{10{,}000 - X}\right)$$

The problem is modelled as:

$$\begin{aligned}
\text{maximize } f(X) &= 6X - \left(8000 + 2X + \frac{X}{10{,}000 - X}\right) \\
\text{subject to} & \\
X &\geq 0 \\
X &\leq 9999
\end{aligned}$$

Solution using the Excel Solver

If we reserve cell A1 for X, then the following formula in cell B1 represents the function: `=6*A1-(8000+2*A1+A1/(10000-A1))`. We can put the 9999 in say cell A2, and then ask the Solver to maximize cell B1, with A1 as the only variable cell, and add the constraint A1 $\leq$ A2. Clicking on the box for non-negativity, and selecting the GRG Nonlinear algorithm, we obtain 9950 in cell A1, and 31,601 in cell B1. Hence by producing 9,950 units per day, we maximize the profit, which will be $31,601.

7.6.4 An Optimal Route for an Oil Pipeline

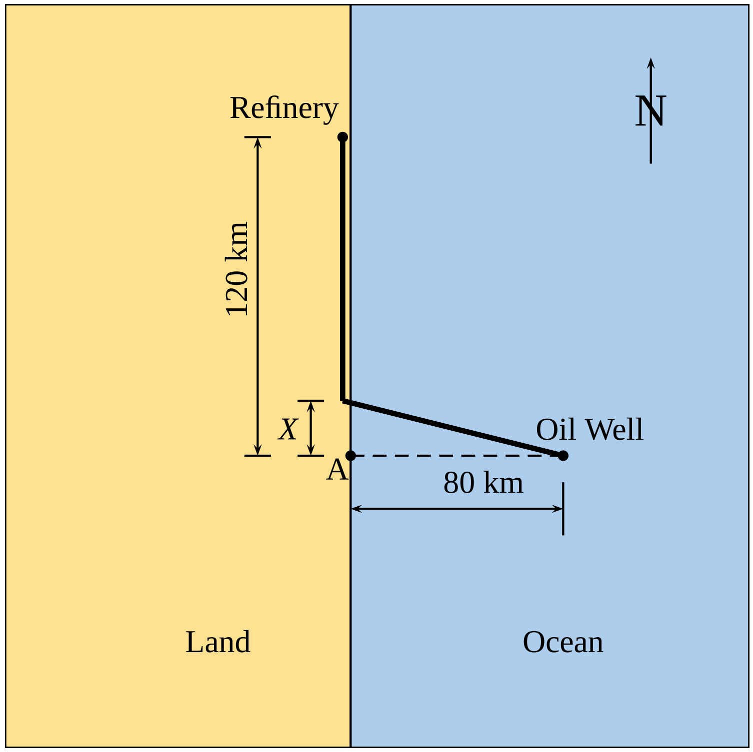
Description

An oil company wishes to build a pipeline from an offshore oil well to a refinery located on the coast. The coastline runs north-south, with the ocean lying to the east. The oil well is located 120 km south and 80 km east of the refinery. The pipeline will be laid straight through the water, will bend at the point at which it comes ashore, and then travel straight along the coast on land to the refinery. It costs $200,000/km to build a pipeline on land; it costs $800,000/km to build a pipeline at sea. [If the costs were the same, they would simply build a pipeline from the well to the refinery through the water.] We wish to determine the location at which the pipeline should come ashore so that the total cost of building the pipeline is minimized.

Formulation

At the outset, it is helpful to draw a picture of the situation. Of course, we cannot accurately mark the spot at which the pipeline comes ashore, since that is the unknown of the problem, but this does not matter.

The locations and distances are as shown in the following diagram, with the proposed route of the pipeline shown as a thick line.



We are seeking the location of the point where the pipeline comes ashore. This can be specified by defining reference point A as a point which is due south of the refinery, and due west of the oil well, and then we are seeking a point which is X km north of point A. [An alternative approach is to let Y run due south from the refinery. Clearly, $X+Y=120$.]

The length of the pipeline on land is $120-X$ km. To find the length at sea, we use the theorem of Pythagoras. On the diagram we can see a triangle with a right angle at the point A, and whose other vertices lie X km north of A (where the pipeline comes ashore) and 80 km east of A (the location of the oil well). The hypotenuse has a length $\sqrt{X^2+80^2}$ km, and is the length of the pipeline at sea.

The cost of the pipeline is the cost on land plus the cost at sea. The cost on land is the per-km cost of \$200,000 multiplied by the number of km (which is $120-X$); the cost at sea is the per-km cost of \$800,000 multiplied by the number of km at sea (which is $\sqrt{X^2+80^2}$). The total cost of the pipeline, in dollars, is

$$f(X) = 200,000(120-X) + 800,000\sqrt{X^2+80^2}$$

Note that we could have written $f(X)$ in thousands of dollars, which would have made the per-km cost coefficients 200 and 800; we could have written $f(X)$ in millions of dollars, which would have made the coefficients 0.2 and 0.8. Whichever we do, we will eventually obtain the same value for X.

Solution using the Excel Solver

Here we use the Excel function **SQRT** for finding the square root of a non-negative number. At a minimum, we want Excel to calculate the optimal values of X and $f(X)$. However, using column A for labels, we can put X into cell B1, the length of the pipeline on land into cell B2 (=120 − B1), and the length of the pipeline at sea into cell B3 (`=SQRT(B1^2+80^2)`). The objective function in cell B5 is `=200000*A2+800000*A3`. Displaying the formulas we have:

	A	B
1	Distance X (km)	
2	Pipeline on land	=120-B1
3	Pipeline at sea	=SQRT(B1^2+80^2)
4		
5	Total Cost	=200000*B2+800000*B3

With B1 ≥ 0 we then ask the Solver to minimize B5 with B1 being the only variable cell. (The values in B2 and B3 will change, but they are not variable cells.) We obtain:

	A	B
1	Distance X (km)	20.65591117
2	Pipeline on land	99.34408883
3	Pipeline at sea	82.62364472
4		
5	Total Cost	$85,967,733.54

7.7 Applications with Multiple Variables

7.7.1 Tunneling in an Underground Mine

A mining company wishes to connect three points lying on the same elevation. Places are referenced by a grid system where (a, b) is a point located a metres east and b metres north of a standard point. The three points are located at (30, 240),

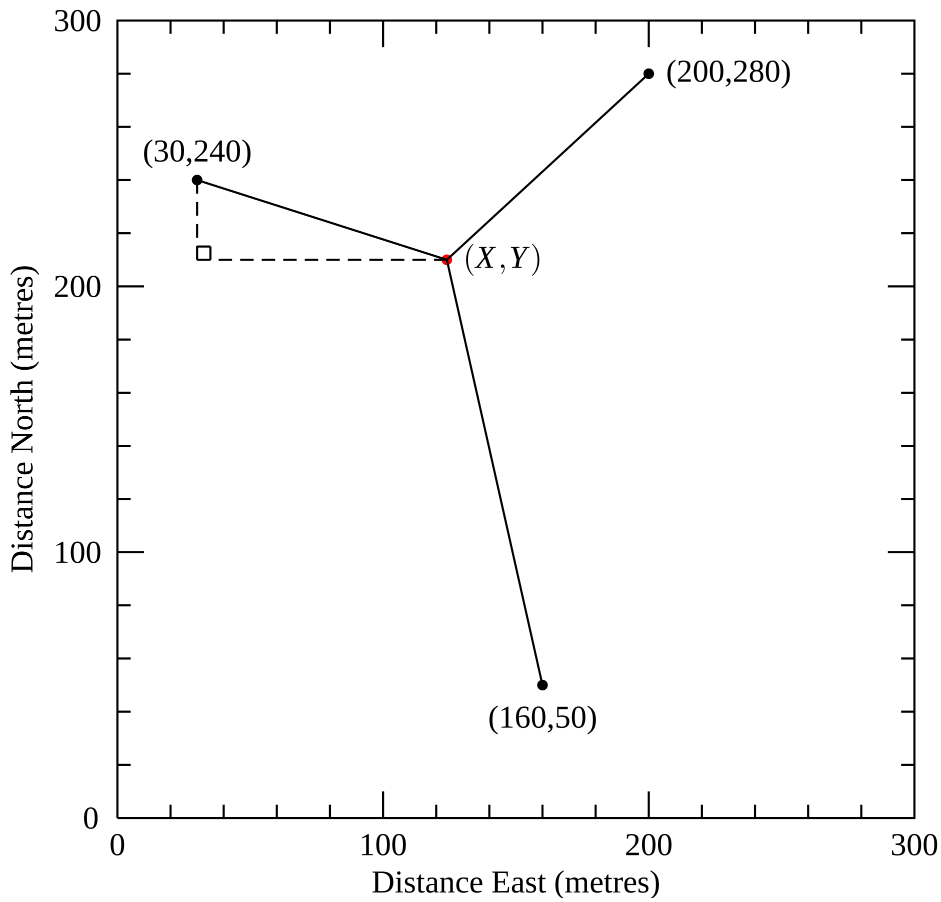


Figure 7.5: Tunnels from Three Points to a Junction Point

(160, 50), and (200, 280). They know that to minimize the construction cost of the tunnels they need to find a point lying in the interior of the triangle defined by the first three points. This point will be the junction point of the three tunnels. They wish to know the location of this point, and the total distance of the tunnels.

First we draw a picture of the situation. We can plot the three defined points, but we have to make an educated guess as to the location of the junction point (X,Y), which of course we seek to determine. The picture is shown in Figure 7.7.1.

Three tunnels need to be made from point (X,Y) to the three given points. Each tunnel can be thought of as the hypotenuse of a right-angled triangle. One of these hypotenuses is shown between Point 1 at (30,240) and the point to be determined, (X,Y); the other two are similar. By using the theorem of Pythagoras the distance to each point from (X,Y) is:

Point	Distance
(30,240)	$\sqrt{(30-X)^2+(240-Y)^2}$
(160,50)	$\sqrt{(160-X)^2+(50-Y)^2}$
(200,280)	$\sqrt{(200-X)^2+(280-Y)^2}$

Therefore the function we wish to minimize is the sum of these three distances. Namely

$$\begin{aligned} f(X,Y) &= \sqrt{(30-X)^2+(240-Y)^2}+\sqrt{(160-X)^2+(50-Y)^2} \\ &+ \sqrt{(200-X)^2+(280-Y)^2} \end{aligned}$$

We will solve this using the Excel Solver. An easy way to write the function is to make it the sum of three tunnel lengths, where, after a writing a formula to calculate the distance from point 1 to (X,Y), we can simply copy the formula for the other two points. Doing this we obtain:

	A	B	C	D	E
1		X	Y		
2					
3					
4	Point 1	30	240	Tunnel 1	=SQRT((B4-$B$2)^2+(C4-$C$2)^2)
5	Point 2	160	50	Tunnel 2	=SQRT((B5-$B$2)^2+(C5-$C$2)^2)
6	Point 3	200	280	Tunnel 3	=SQRT((B6-$B$2)^2+(C6-$C$2)^2)
7					
8				Total	=SUM(E4:E6)

Invoking the GRG Nonlinear algorithm on the Solver, in which we minimize cell E8 with the variable cells being B2:C2, we obtain:

	A	B	C	D	E
1		X	Y		
2		124.2374	210.4101		
3					
4	Point 1	30	240	Tunnel 1	98.7738
5	Point 2	160	50	Tunnel 2	164.3483
6	Point 3	200	280	Tunnel 3	102.8724
7					
8				Total	365.9944

We find the optimal location to be at about (124.24, 210.41) with a total distance of 365.99 metres.

This type of problem is called a ***Steiner tree*** problem. The three point problem has an easy geometric solution. See https://en.wikipedia.org/wiki/Steiner_tree_problem.

7.7.2 An Example from Finance

The manager of a portfolio can place the money in the shares of publicly traded companies, in a money-market fund, or in the government bond market. Over the past several years, the average returns have been 1.5%, 0.8%, and 1.1%, respectively. They know the covariance of the returns, which, in the order shares/money-market/bonds is:

$$\begin{pmatrix} 4.2 & 1.7 & 1.4 \\ 1.7 & 0.8 & 0.6 \\ 1.4 & 0.6 & 0.5 \end{pmatrix}$$

We use the symbol $\mathbf{C}$ for this covariance matrix. The manager wants to form a portfolio of minimum risk, with an expected return of at least 1.2%. We define X_i to represent the fraction of the portfolio invested in shares ($i = 1$), the money-market ($i = 2$), or government bonds ($i = 3$). The objective is to minimize:

$$\mathbf{XCX}^T = (X_1, X_2, X_3) \begin{pmatrix} 4.2 & 1.7 & 1.4 \\ 1.7 & 0.8 & 0.6 \\ 1.4 & 0.6 & 0.5 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

We could expand this by hand, using matrix multiplication twice, first to find $\mathbf{CX}^T$, and then again to find $\mathbf{XCX}^T$. Doing this we find

$$f(X_1, X_2, X_3) = 4.2X_1^2 + 0.8X_2^2 + 0.5X_3^2 + 2(1.7)X_1X_2 + 2(1.4)X_1X_3 + 2(0.6)X_2X_3$$

However, rather than doing this by hand we can ask Excel to do the matrix multiplication for us. The weights must sum to 1, hence the entire model is:

$$\begin{array}{rrcl}
\text{minimize} & \mathbf{XCX}^T & & \\
\text{subject to} & & & \\
\text{All invested} & X_1 + X_2 + X_3 & = & 1 \\
\text{Return} & 1.5X_1 + 0.8X_2 + 1.1X_3 & \geq & 1.2 \\
 & & & \\
 & X_1, X_2, X_3 \geq 0 & &
\end{array}$$

Using the Solver

This example is put into Excel, with the range C2:E2 reserved for the values of the variables, and cell A3 reserved for the objective function value. Because the constraints are linear, we can calculate the numerical value of the left-hand side of each constraint as we did before, using the SUMPRODUCT function. However, the objective function is too complicated for this approach. There are two possible approaches

1. We could enter the expression

   $f(X_1, X_2, X_3) = 4.2X_1^2 + 0.8X_2^2 + 0.5X_3^2 + 3.4X_1X_2 + 2.8X_1X_3 + 1.2X_2X_3$

   into cell A3 as

   ```
   =4.2*C2^2+0.8*D2^2+0.5*E2^2+3.4*C2*D2+2.8*C2*E2+1.2*D2*E2
   ```

   This approach is simple for this little example, but had there been say ten possible investments calculating all these terms would have been tedious and prone to error.

2. The other approach is to enter the covariance matrix into Excel, and then let Excel find $\mathbf{XCX}^T$. To do this we need to use the TRANSPOSE command, and then use the MMULT command twice (see Chapter 1 for information about these commands).

	A	B	C	D	E	F	G	H
1			X1	X2	X3			
2	OFV		0	0	0			
3	=MMULT(C2:E2,G3:G5)		4.2	1.7	1.4	=TRANSPOSE(C2:E2)	=MMULT(C3:E5,F3:F5)	
4			=D3	0.8	0.6	=TRANSPOSE(C2:E2)	=MMULT(C3:E5,F3:F5)	
5			=E3	=E4	0.5	=TRANSPOSE(C2:E2)	=MMULT(C3:E5,F3:F5)	
6	Constraints							
7	All Invested		1	1	1	=SUMPRODUCT($C$2:$E$2,C7:E7)	=	1
8	Return		1.5	0.8	1.1	=SUMPRODUCT($C$2:$E$2,C8:E8)	>=	1.2

In using the Solver, we need to click on "Make Unconstrained Variables Non-Negative", and set the solving method to "GRG Nonlinear". Solving we obtain:

	A	B	C	D	E	F	G	H
1			X1	X2	X3			
2	OFV		0.25	0	0.75			
3	1.06875		4.2	1.7	1.4	0.25	2.1	
4			1.7	0.8	0.6	0	0.875	
5			1.4	0.6	0.5	0.75	0.725	
6	Constraints							
7	All Invested		1	1	1	1	=	1
8	Return		1.5	0.8	1.1	1.2	>=	1.2

The recommendation is for the manager to invest 25% of the portfolio in shares of publicly-traded companies, and to invest the other 75% in government bonds.

7.8 Summary

Problems with multiple objectives can be modeled using deviational variables. All goal programming models involve minimization. One approach is to weight the deviational variables using a single objective function. We then obtain a single linear optimization problem, which we can solve using the Excel Solver. The other approach is preemptive goal programming, in which the goals are ranked in order of importance. Starting with the most important goal, a sequence of linear programming models is solved. In each, we are minimizing one deviational variable at a time, subject to all system constraints, and all constraints associated with higher-ranked goals.

We modeled a simple inventory system, which was solved using analytical calculus to obtain the EOQ formula, We then saw the use of the GRG algorithm in the Excel Solver, which can not only solve any numerical problem involving a single variable, but can also handle multiple variables and constraints.

We examined a variety of applications of single-variable differential calculus. We went on to consider some problems which cannot be reduced to a single variable.

7.9 Problems for Student Completion

7.9.1 Restaurant Location

This example illustrates non-preemptive goal programming.

Some entrepreneurs wish to establish a restaurant in a central location so that it will serve three suburbs. On a rectangular grid with axes labelled X_1 (horizontal) and X_2 (vertical), the centres of the three suburbs are located at (1,2), (6,18), and (12,8). All roads run east-west (parallel with the X_1 axis) or north-south (parallel with the X_2 axis). The distance between any two points is therefore rectilinear. For example, the distance between the centres of suburbs 2 and 3 is $\mid 12-6 \mid + \mid 8-18 \mid = 6+10 = 16$.

(a) Formulate the restaurant location problem as a goal programming model.

(b) Use the Excel Solver to determine the best location for the restaurant.

(c) Now suppose that suburb 1 is twice as large as suburb 2, and three times as large as suburb 3. Write the new model, and solve it using the Excel Solver.

7.9.2 Moose Licences

This example illustrates preemptive goal programming.

A licence to shoot a moose costs \$100 for residents and \$800 for non-residents. The government must decided how many licences to issue in both categories. Demand for resident licences appears to be unlimited; indeed, they would often have to hold a lottery to decide who received one. Demand for non-resident licences is about 10,000 per year. The government will not issue more than 35,000 moose licences per year. At least 75% of all licences must go to residents.

(a) If the objective is to maximize revenue, find the optimal solution graphically.

(b) Now suppose that the government wants annual revenues from the sale of moose licences to be at least \$10,000,100. Verify that there is no feasible solution.

(c) Now suppose that the demand for up to 10,000 non-resident licences is part of a system constraint. The goal priorities in descending order of importance are (i) earn at least \$10,000,100 in revenue (ii) issue at least 75% of licences to residents, and (iii) limit the licences to 35,000. Solve the revised problem graphically.

7.9.3 Admission Prices

A museum charges \$8 per person for admission. For each visitor, there is a cost of \$3 for cleaning, insurance, security, and so on. In addition, there is an annual overhead cost of \$460,000. Currently 90,000 people per year visit the museum.

(a) How much money is the museum losing each year?

The Board of Governors is considering a change in the admission price. For each \$1 increase/decrease to the price, the number of visitors per annum will go down/up by 12,500.

(b) What admission price maximizes the profit, and what is this optimal profit?

(c) Suppose that the admission charge must, for practical reasons, be priced using dollars and quarters only. What are the optimal price and profit now?

7.9.4 Rescue in the Water

A lifeguard is watching over a beach. The lifeguard is in a tower located on the shoreline (which runs east-west), and sees a person in trouble in the water. The victim is 90 metres from the closest point on the shoreline, and this point on the shoreline is 170 metres east of the tower.

The lifeguard can run along the beach at a rate of 8 metres per second, and can swim in the water at a rate of 3 metres per second. The lifeguard must choose a point on the shoreline to which he will run, and from which he will swim to the victim, so that the total time to reach the victim is minimized.

(a) Draw a picture of this situation.

(b) Find the lifeguard's optimal route, and the optimal time.

7.9.5 Inventory

Note: This question requires analytical calculus; it cannot be done using the Excel Solver.

In an inventory system the warehouse must be as large as the maximum amount of the inventory. If the warehouse is the predominant cost, then it may be appropriate to say that the cost of holding inventory should be based not on the average inventory level, but on the maximum inventory level. Based on this modification to the standard EOQ model, derive a modified EOQ formula.

7.9.6 Tunneling in an Underground Mine

A mining company wishes to connect three points lying on the same elevation. Places are referenced by a grid system where (a, b) is a point located a metres east and b metres north of a standard point. The three points are located at (110, 250), (190, 120), and (230, 270). They know that to minimize the construction cost of the tunnels they need to find a point lying in the interior of the triangle defined by the first three points. This point will be the junction point of the three tunnels. They wish to know the location of this point, and the total distance of the tunnels.

(a) Draw a picture of this situation, showing hypothetically where the junction point might be placed, and draw lines to represent the tunnels.

(b) Formulate this problem.

(c) Find the numerical solution using the Excel Solver.

7.9.7 Asset Allocation

A wealthy couple have three children named Xena, Yuri, and Zoe. To give their children a lesson in entrepreneurship, the parents have decided to invest a total of \$35,000. They asked their children what they could accomplish if they were given some of the money. Xena said, "Whatever you give me, I will return not only the principal but the square root of the principal as well." (For example, if she were given \$1600, she would return $1600 + \sqrt{1600} = 1640$ dollars.) Yuri thought that he could do better than his younger sister: "I'll return the principal plus twice the square root of the principal", he boasted. Their older sister Zoe felt that she had to do even better: "I'll return the principal plus three times the square root of the principal." The parents wonder how the \$35,000 should be distributed to their children, so as to maximize the total net return.

(a) Formulate a model for this problem.

(b) Obtain the solution using the Solver.

Chapter 8

Decision Analysis I

Decision modeling so far in this course has been in a deterministic context. Now we present some ways of modeling and solving problems which involve probabilities. Knowing the basic concepts of probability is required; these are explained in Appendix E.1.

8.1 Payoff Matrices

8.1.1 Introduction

The simplest situation involving decision making under uncertainty has the following attributes:

- There is one decision; the decision maker must choose one of several alternatives.
- There is one event; one of several possible outcomes will occur.
- For each combination of alternative and outcome we can calculate the payoff (which may be negative) to the decision maker.

The order is very important: *the decision must precede the event*. First an alternative is chosen, and then an outcome occurs. Here are some common examples:

1. At 8 a.m. you must decide whether or not to carry an umbrella; later that day you find out whether or not it rains.

2. Before a hockey game, you decide whether or not to place a bet on the outcome; and at the end of the game you find out which team has won.

8.1.2 Example

Problem Description

An amateur theatre company wishes to mount a play. A three night run is planned, and a particular play has been chosen. They have already spent or have committed to spend $2500 for such things as costumes, makeup, royalties to the copyright owners, and so on. They are definitely going ahead with the play; the only decision they must make is where to hold it. Small, medium, and large theatres are available for rent which hold 100, 400, and 1200 people respectively. Three nights rent at each theatre would cost $600, $1800, and $4700 respectively. They must make a commitment to one of these theatres several weeks before the run begins.

The theatre company has already decided to price all the tickets at $10.00 each.[1] Because everyone in the theatre company is a volunteer, they can price the tickets at an affordable price. All they care about from a financial point of view is to at least cover their expenses over the long term.

The demand for the play is uncertain until the run begins. Demand is heavily influenced by the critics' reviews. The critics will attend a dress rehearsal the night before the first performance, and their opinions will be printed and broadcast in the media the next morning.

The directors of the company know from experience that demand for plays falls into four broad categories of interest: fringe; average; great; and heavy. We will assume that the demand is spread equally across the three nights. The total number of people who wish to see a play over a three-night run is typically 250 for fringe, 800 for average, 2300 for great, and 4500 for heavy.

These are demand levels, not necessarily the number of tickets sold. For example, if a play sells every seat in a 250 seat theatre for three nights, and if another 50 people were wait-listed for tickets but could not obtain them, then 750 tickets were sold, but the demand was for 800 tickets.

The demand is an event in which one of four outcomes will occur. To estimate

[1] To keep things simple, we often ignore taxes when developing models for educational purposes. If you want to think of ticket prices as including taxes imposed by the government, then imagine that we are dealing with the net after taxes revenue, for example the tickets could be $12.50 each and from this they must pay $2.50 per ticket sold in taxes, leaving them with a net revenue of $10 per ticket sold.

the probabilities of these four outcomes, the theatre company could look at the historical data for plays of this type with tickets sold in this price range. Suppose that of one hundred plays in the past, the interest attracted was twenty for fringe, seventy for average, nine for great, and one for heavy. We would then estimate the chance of the next play attracting fringe interest as

$$P(\text{fringe interest}) = \frac{20}{100} = 0.20$$

Continuing in this manner we would estimate the probabilities for average, great, and heavy as 0.70, 0.09, and 0.01 respectively.

Using historical data to estimate probabilities ignores such factors as changing consumer tastes and economic conditions, but we have to start somewhere. Using these numbers we will obtain one conclusion after solving the model, but another set of numbers will often lead to a different conclusion.

This model has been kept simple in that everything has been decided except one thing – which theatre to rent. This is the problem which we shall now solve.

Model Formulation

In all models with decision making under uncertainty, we must define the decisions, their alternatives, the events, and their outcomes. Some textbooks stress the use of symbols for this purpose, however another approach is to use words only, and then define a shortcut word to use in place of each longer phrase.

For both approaches we have the following:

They must decide where to hold the play. The alternatives are to rent a small theatre with 100 seats, or rent a medium-sized theatre with 400 seats, or rent a large theatre with 1200 seats. The event is the demand for tickets. The possible outcomes are as follows: there is fringe interest with demand for 250 tickets; there is average interest with demand for 800 tickets; there is great interest with demand for 2300 tickets; or there is heavy interest with demand for 4500 tickets.

Because it takes a lot of space to write all these words every time we wish to refer to them, we need a shortcut form. In the method of using symbols, the decision is symbolized with the letter D, and the three alternatives have subscripts on the letter A, making them A_1, A_2, and A_3. The event is symbolized with the letter E, and its four outcomes have subscripts on the letter O, making them O_1, O_2, O_3, and O_4.

The alternative and outcome symbols mean the following:

	Alternative	Cost
A_1	rent a small theatre with 100 seats	\$600
A_2	rent a medium-sized theatre with 400 seats	\$1800
A_3	rent a large theatre with 1200 seats	\$4700
	Outcome	**Probability**
O_1	there is fringe interest; the demand is for 250 tickets	0.20
O_2	there is average interest; the demand is for 800 tickets	0.70
O_3	there is great interest; the demand is for 2300 tickets	0.09
O_4	there is heavy interest; the demand is for 4500 tickets	0.01

The other approach is to use one word (or a very short phrase) to mean the entire long phrase. Such words must be unique. For example, we cannot use "medium" to refer to both a medium-sized theatre and to average interest. Using this approach we could use the following words:

	Alternative	Cost
small	rent a small theatre with 100 seats	\$600
medium	rent a medium-sized theatre with 400 seats	\$1800
large	rent a large theatre with 1200 seats	\$4700
	Outcome	**Probability**
fringe	there is fringe interest; the demand is for 250 tickets	0.20
average	there is average interest; the demand is for 800 tickets	0.70
great	there is great interest; the demand is for 2300 tickets	0.09
heavy	there is heavy interest; the demand is for 4500 tickets	0.01

Whichever method is used, the important thing is that the person making the model must understand what the alternatives and outcomes are.

The only other pieces of information we need from the problem description is that the revenue is \$10 per ticket sold, and that the play runs for three nights. The other expenses such as costumes, makeup, royalties to the copyright owners, and so on are what are called *sunk costs*. A sunk cost is money which is either already spent or has already been committed, and is therefore irrelevant to the decision. Indeed, even if these fixed expenses (which total \$2500) were not already committed, they would not affect the decision in this example, because all alternatives would contain these same expenses.

Model Solution

There are three alternatives, and four outcomes, hence there are three times four equals twelve situations which need to be evaluated. First we see what happens if a small theatre is rented, and the play only attracts fringe interest. The 100-seat small theatre can hold 300 people over three nights, but only 250 people want to see the play, so only 250 tickets are sold. The net revenue from the ticket sales is therefore \$10 times 250 = \$2500. We can now find what is often called the "profit", but we define a new term *payoff*, which can mean profit, cost, or revenue depending on the context. The payoff is found by subtracting the \$600 rent from the \$2500 from the sales of tickets, i.e. \$1900.

If a small theatre is rented, but the demand turns out to be average, then there are more willing customers (800) than there are seats (300). The number of ticket sales is therefore just 300. For any situation, we can say that the number of tickets sold is the capacity of the theatre (over three nights), or the demand for tickets, whichever is *less*. The payoff is

$$\$10(300) - \$600 = \$2400$$

We do not need to analyze in detail what happens if more potential customers (great or heavy) show up when only a small theatre has been rented; no more tickets can be sold, so the payoff will remain at \$2400.

Now suppose that a medium-sized theatre is rented at a cost of \$1800. With a 400 seat capacity, a three-night run gives a maximum sales capacity of 1200 tickets. There's plenty of space with fringe or average demand, but the capacity of 1200 is reached with great or heavy demand. With fringe interest the payoff is:

$$\$10(250) - \$1800 = \$700$$

With average interest the payoff is:

$$\$10(800) - \$1800 = \$6200$$

With either great or heavy demand the payoff is

$$\$10(1200) - \$1800 = \$10,200$$

If a large theatre with 1200 seats is rented for \$4700, the three-night capacity is 3600 people. This is sufficient for all but heavy demand. The number of tickets sold will equal the demand if interest is fringe, average, or great, and will equal the total capacity (3600) if there is heavy demand. Hence we have:

Outcome	fringe	average	great	heavy
3-Night Capacity	3600	3600	3600	3600
Demand	250	800	2300	4500
# of Tickets Sold	250	800	2300	3600
Net Ticket Revenue	\$2500	\$8000	\$23,000	\$36,000
Rent	\$4700	\$4700	\$4700	\$4700
Payoff	−\$2200	\$3300	\$18,300	\$31,300

The preceding calculations do not need to be always explicitly written out as we have done here. Often the calculations can be done on a calculator, with just the final payoffs being written down. Or, as we soon shall see, we can use a spreadsheet to do the calculations. Of course, to do this by any means we must understand how the final payoff is derived. In all twelve cases, the payoff is computed as:

$$
\begin{aligned}
\text{payoff} &= \text{ticket revenue} - \text{rent} \\
&= \text{ticket price} \times \text{number of tickets sold} - \text{rent} \\
&= \text{ticket price} \times \min\{\text{ three-night capacity, demand }\} - \text{rent}
\end{aligned}
$$

All of this information can be conveniently summarized in what is called a *payoff matrix* (also called a payoff table). In doing this by hand, just one payoff matrix is drawn. However, to help explain it, we draw it once with just the borders, then with the main body filled in, and then with the right-hand side filled in.

In the main body of the payoff matrix, each row represents an alternative, and each column represents an outcome. Labels for the alternatives appear on the left-hand side, and labels for the outcomes appear on the top. The final row lists the probabilities of the outcomes. The final column is reserved for the *expected value* of each alternative – this will be explained shortly.

It is helpful if we put the theatre capacity (over three nights) and the cost of the rent next to the name of the alternative, and the demand as a number next to the names for the four levels of demand. Doing this the payoff matrix begins as:

			Demand for Tickets				
Theatre Size	3-Night Capacity	Rent	Fringe 250	Average 800	Great 2300	Heavy 4500	Expected Value
Small	300	\$600					
Medium	1200	\$1800					
Large	3600	\$4700					
		Probability	0.20	0.70	0.09	0.01	

Using the formula "=ticket price $\times$ min$\{$ three-night capacity, demand $\}$ $-$ rent", each payoff is calculated and put into the table. If we are doing these calculations using a calculator, we would look for shortcuts like noticing the repetition of the "2400" for the first alternative.

We of course have already done these calculations by hand, and hence we have (dropping the dollar signs):

Theatre Size	3-Night Capacity	Rent	Demand for Tickets: Fringe 250	Average 800	Great 2300	Heavy 4500	Expected Value
Small	300	\$600	1900	2400	2400	2400	
Medium	1200	\$1800	700	6200	10,200	10,200	
Large	3600	\$4700	−2200	3300	18,300	31,300	
		Probability	0.20	0.70	0.09	0.01	

If we wish to use a spreadsheet, we will input the theatre size, and let the 3-night capacity be found as part of the formula, which is entered once and then is copied. Besides doing the calculations, using a spreadsheet makes it easy to change colours and/or fonts to highlight information. The real advantage, however, is that it easily allows us to see what happens when some of the information is changed.

In spreadsheet form we begin with:

	A	B	C	D	E	F	G	H
1			Price	Demand for Tickets				
2	Theatre	Number	\$10	Fringe	Average	Great	Heavy	Expected
3	Size	of Seats	Rent	250	800	2300	4500	Value
4	Small	100	\$600					
5	Medium	400	\$1,800					
6	Large	1200	\$4,700					
7			Probability	0.20	0.70	0.09	0.01	

We want to make a formula in cell D4 which we can copy to the range D4:G6. The number of seats available is in cell B4, hence the 3-night capacity is `3*B4`. The demand is in cell D3, and the cost of the rent is in cell C4. For some of the copied cells, we need to use an absolute rather than a relative cell address, which is accomplished by placing a dollar sign in front of the column or row which needs to be frozen. Hence we must use a dollar sign to freeze the 'B' in 'B4', the '3' in 'D3', and the 'C' in 'C4'. For cell C2, which contains the ticket price, we need a dollar sign in front of both the C and the 2 to freeze both the column and the row. The formula to be placed in cell D4 is therefore:

```
=$C$2*MIN(3*$B4,D$3)-$C4
```

With the numbers in the main body of the payoff matrix being calculated by the spreadsheet (commas will not appear unless special formatting is used) we have:

	A	B	C	D	E	F	G	H
1			Price	Demand for Tickets				
2	Theatre	Number	$10	Fringe	Average	Great	Heavy	Expected
3	Size	of Seats	Rent	250	800	2300	4500	Value
4	Small	100	$600	1,900	2,400	2,400	2,400	
5	Medium	400	$1,800	700	6,200	10,200	10,200	
6	Large	1200	$4,700	-2,200	3,300	18,300	31,300	
7			Probability	0.20	0.70	0.09	0.01	

There is an *expected value* associated with each alternative. Recall from having studied random variables that in general, if there are n outcomes, and the probability of outcome i is p_i, and the payoff of outcome i is x_i, then the expected value is defined as:

$$E(\mathbf{X}) = \sum_{i=1}^{n} p_i x_i \tag{8.1}$$

The current example has four outcomes. The expected value associated with renting a large theatre is

$$\begin{aligned} \text{EV(large)} &= 0.20(-2200) + 0.70(3300) + 0.09(18,300) + 0.01(31,300) \\ &= -440 + 2310 + 1647 + 313 \\ &= 3830 \end{aligned}$$

What this figure means is that if the theatre company were to face the same situation many times, and if they were to choose a large theatre each time, then over time their profits/losses would average out to \$3,830. The actual payoff on a particular play will be either −\$2200, or \$3300, or \$18,300, or \$31,300. Hence the expected value is none of the actual values; it is simply a long-term average value.

When some of the outcomes are the same, as occurs for the medium-sized theatre alternative, we can factor the numbers if we wish:

$$\begin{aligned}\text{EV(medium)} &= 0.20(700)+0.70(6200)+(0.09+0.01)(10,200)\\ &= 140+4340+1020\\ &= 5500\end{aligned}$$

The small theatre alternative is even easier:

$$\begin{aligned}\text{EV(small)} &= 0.20(1900)+(0.70+0.09+0.01)(2400)\\ &= 380+1920\\ &= 2300\end{aligned}$$

We have shown these calculations in detail because the material is new, but from now on we will simply calculate the numbers and write only the final answer. Filling in the numbers in the Expected Value column we have:

			Demand for Tickets				
Theatre Size	3-Night Capacity	Rent	Fringe 250	Average 800	Great 2300	Heavy 4500	Expected Value
Small	300	\$600	1900	2400	2400	2400	\$2300
Medium	1200	\$1800	700	6200	10,200	10,200	\$5500
Large	3600	\$4700	−2200	3300	18,300	31,300	\$3830
		Probability	0.20	0.70	0.09	0.01	

Now let us see how to do this using a spreadsheet. In cell H4, we wish to write a formula which will find the "dot product" of the probabilities in D7:G7 with the payoffs in D4:G4. One way to do this (ignoring absolute cell addresses for the moment) is:

```
=D7*D4+E7*E4+F7*F4+G7*G4
```

Because we only have four outcomes, we could do it this way. However, this approach would be very cumbersome if we had say twenty outcomes. Therefore, we will instead use the spreadsheet SUMPRODUCT function.

The SUMPRODUCT function finds the dot product of the numbers in range1 with the numbers in range2, where both ranges are rows (or columns) of equal size. The syntax is `SUMPRODUCT(range1,range2)`. We must put absolute cell addresses on row 7 (the probabilities), hence the formula to be placed in cell H4 is:

```
=SUMPRODUCT(D$7:G$7,D4:G4)
```

This formula is copied into cells H5 and H6. We obtain:

	A	B	C	D	E	F	G	H	I
1			Price	Demand for Tickets					
2	Theatre	Number	$10	Fringe	Average	Great	Heavy	Expected	
3	Size	of Seats	Rent	250	800	2300	4500	Value	
4	Small	100	$600	1,900	2,400	2,400	2,400	2,300	
5	Medium	400	$1,800	700	6,200	10,200	10,200	5,500	Best
6	Large	1200	$4,700	-2,200	3,300	18,300	31,300	3,830	
7			Probability	0.20	0.70	0.09	0.01		

The formatting of the numbers in the range H4:H6 is a matter of individual preference. For example, any of 2300, $2300, or $2300.00 could be used.

On average, the best alternative is the one with the highest expected value. In the next section, we shall look at alternate decision criteria, but in the absence of reason to the contrary the preferred criterion for decision making under uncertainty will be to choose the alternative with the highest expected value.

Recommendation

For the example at hand, the best alternative is clearly to rent a medium-sized theatre, with an expected payoff of $5500. As we said in the introductory section, the developer of the model must make the recommendation clear to the customer of the model. In this example, the customer is the theatre company. They might not be familiar with payoff matrices or spreadsheets, so we focus on giving the recommendation - the spreadsheet itself is just an appendix. For the sake of this course, let's say that the term "expected payoff" can be used; in real life more explanation would be required. Hence within this course we would write the recommendation as:

Recommendation Rent a medium-sized theatre, with an expected payoff of $5500 before the deduction of $2500 in fixed expenses, or $3000 after making this deduction.

However, in giving a recommendation to the theatre company in real-life, something along the following lines might be appropriate:

To:	The Management Committee, Amateur Theatre Group
From:	J. Blow, Decision Modeling Consulting Company
Subject:	Theatre Rental

Thank you for this opportunity to assist your theatre company, which I am happy to provide on a *pro-bono* basis. After studying the three alternatives, I conclude that renting a medium-sized theatre would be best. Based on the assumptions which you provided, the profit (before deducting fixed expenses such as costumes, makeup, and royalties to the copyright owners) will be either $700, $6200, or $10,200; if a situation like this were to be repeated many times the profit would average out to $5500. After deducting the $2500 in fixed expenses the company will be left with a profit (loss) of ($1800), $3700, or $7700; if a situation like this were to be repeated many times the profit would average out to $3000. A spreadsheet which I used to make the gross profit calculations appears as an appendix to this memo.

J. Blow

Analyst

	A	B	C	D	E	F	G	H	I
1			Price	Demand for Tickets					
2	Theatre	Number	$10	Fringe	Average	Great	Heavy	Expected	
3	Size	of Seats	Rent	250	800	2300	4500	Value	
4	Small	100	$600	1,900	2,400	2,400	2,400	2,300	
5	Medium	400	$1,800	700	6,200	10,200	10,200	5,500	Best
6	Large	1200	$4,700	-2,200	3,300	18,300	31,300	3,830	
7			Probability	0.20	0.70	0.09	0.01		

8.1.3 Salvage Value

In this section we consider an extension to the basic model of decision making under uncertainty when there is one decision and one event. First, we introduce the concept of a *salvage value*, which is the remaining value of something which has not sold at the regular price. It could also be called a *clearance price*. It is often used when a company needs to clear inventory quickly; here are some examples:

1. A newspaper has a regular price of $1.75. The next morning, the left-over copies are sold to a paper recycling operation for 5 cents each.
2. A winter coat is priced at $300. If it's not sold by the end of March, it's priced to clear at $160.
3. A hardcover book lists for $39.95. Some people buy it at this price, but when sales drop to nothing, the book is priced to clear at $9.99.

Sometimes items for sale pass through multiple price levels. For example, a DVD of a recent release may be priced as high as $34.99, but then the price is progressively lowered to $19.99, then $12.99, and finally the product is priced to clear at $5.00. However, we will not make models with more than two price levels, for this only makes the problem complex. Also, unless stated to the contrary, we will assume that all the inventory which remains after trying to sell the product at the regular price can in fact be sold at the salvage value. Another assumption is that the existence of a clearance price does not affect the regular sales. The solution to the model depends on the assumptions made – if the assumptions are unrealistic, then so too will be the "solution".

Theatre Example with Salvage Value

Suppose that fifteen minutes before showtime, the theatre company decides to price all unsold seats at $2.00 each.[2] A sign is placed outside the theatre announcing the price reduction, and hopefully bargain-hunters and passers-by who see the sign will pay the reduced price to see the play. We will begin by investigating what happens if we make the following assumptions:

1. All seats not sold at the regular price will sell-out at the reduced price.

[2] As before, if you wish to think of taxes, imagine that this is the net revenue per ticket after remitting tax to the government.

2. The demand at the regular price is not affected by the existence of the cheaper tickets.

Because we have already solved the problem without the salvage revenue, all we need do is find what the salvage revenue will be in each of the twelve situations (3 alternatives; 4 outcomes) and add it to the previously found payoff in that situation. Clearly, the sell-out situations are unchanged. These are: small theatre with average, great, or heavy demand; a medium-sized theatre with great or heavy demand; and a large theatre with heavy demand. For the non-sellout situations, the salvage revenue is:

$$\$2 \times (\text{three-night capacity} - \text{the demand for tickets at the regular price})$$

Using this formula we obtain:

		Fringe 250	Average 800	Great 2300	Heavy 4500
Small	300	2(300 – 250) = 100	–	–	–
Medium	1200	2(1200 – 250) = 1900	2(1200 – 800) = 800	–	–
Large	3600	2(3600 – 250) = 6700	2(3600 – 800) = 5600	2(3600 – 2300) = 2600	–

Before proceeding further we should question whether these results seem reasonable. The extreme situation is when a large theatre has been rented, but the play only attracts fringe interest. According to the above model, 250 people pay the regular price, and then ten minutes before showtime 3350 people (spread over three nights) arrive to fill the theatre. This is clearly not reasonable. First of all, not that many people would walk by the theatre to obtain tickets, especially a play which has been panned by the critics. A new assumption about demand is therefore required. Perhaps the demand for last-minute discount tickets would only be about 100 tickets per night (300 in total). Another problem is the ability of the ticket office to handle a large volume of last-minute tickets. Even based on a cash-based exact change model, it would be a stretch to think that more than 250 people per night (750 in total) could be admitted this way.

The advantage of working with a model is that it lets us try out more than one possibility. Let's see what happens using the limit of 300 last-minute tickets. We can later see what happens with selling up to 750 last-minute tickets, which would only require us to change one cell in the spreadsheet.

Based on a maximum of 300 last-minute tickets would limit the salvage revenue to a maximum of \$2(300) = \$600. With this assumption the table becomes:

		Fringe 250	Average 800	Great 2300	Heavy 4500
Small	300	100	–	–	–
Medium	1200	600	600	–	–
Large	3600	600	600	600	–

If we now believe that this table seems reasonable we can proceed to the next step, which is to add these payoffs to those obtained before. Doing this, and then finding the new expected values, we obtain:

			Demand for Tickets				
Theatre Size	3-Night Capacity	Rent	Fringe 250	Average 800	Great 2300	Heavy 4500	Expected Value
Small	300	\$600	2000	2400	2400	2400	\$2320
Medium	1200	\$1800	1300	6800	10,200	10,200	\$6040
Large	3600	\$4700	−1600	3900	18,900	31,300	\$4424
		Probability	0.20	0.70	0.09	0.01	

As an aside, we note that there are two ways to find the new expected values. Using the alternative of renting a large theatre to illustrate, one way is to calculate:

$$0.20(-1600)+0.70(3900)+0.09(18,900)+0.01(31,300)=4424$$

The other way is to note that the previous EV in this row was 3830. We added 600 to each of the first three columns, therefore the new EV is:

$$3830+(0.20+0.70+0.09)600=4424$$

With the assumption that the sales of discount tickets are limited to 300, we see that while each of the three EVs changes, the optimal alternative remains the same, i.e. rent a medium-sized theatre.

To do these calculations on a spreadsheet, we modify what we did earlier (page 346). The formula in cell D4 is currently:

```
=$C$2*MIN(3*$B4,D$3)-$C4
```

The number of unsold seats is either 0 or `3*$B4-D$3`, whichever is greater. This is represented as `MAX(0,3*$B4-D$3)`. By our assumption that we cannot sell more than 300 discount tickets, the number of discount tickets sold is either 300, or `MAX(0,3*$B4-D$3)`, whichever is fewer. Hence the number of discount tickets sold is

```
MIN(300,MAX(0,3*$B4-D$3))
```

However, it is better spreadsheet design to put the 300 into a cell, and then let the formula reference this cell. This is because it makes the 300 transparent, and because it makes the 300 easier to change. Suppose that we put the 300 into cell D8. Hence the number of discount tickets sold is:

```
MIN($D$8,MAX(0,3*$B4-D$3))
```

They net $2 for each ticket. Again, good spreadsheet design says that we should put the $2 figure into its own cell, say H8. Hence the salvage revenue is

```
$H$8*MIN($D$8,MAX(0,3*$B4-D$3))
```

Adding this revenue the formula in cell D4 becomes:

```
=$C$2*MIN(3*$B4,D$3)+$H$8*MIN($D$8,MAX(0,3*$B4-D$3))-$C4
```

There may be more than one way to correctly write a formula. For example, if the demand equals or exceeds the number of seats, then the revenue is the number of seats multiplied by $10, otherwise the revenue is the demand multiplied by $10, plus $2 for each seat not sold at the regular price up to a maximum demand of 300 at the lower price. This logic is captured in the following IF statement for cell D4, from which the rent is subtracted:

```
=IF(D$3>=3*$B4,$C$2*3*$B4,$C$2*D$3+$H$8*MIN($D$8,3*$B4-D$3))-$C4
```

This alternate formula is no shorter, but it may be easier to understand. When it is entered into the spreadsheet and copied into the range D4:G6, column D in formula display mode appears as:

	D
1	
2	Fringe
3	250
4	=IF(D$3>=3*$B4,$C$2*3*$B4,$C$2*D$3+$H$8*MIN($D$8,3*$B4-D$3))-$C4
5	=IF(D$3>=3*$B5,$C$2*3*$B5,$C$2*D$3+$H$8*MIN($D$8,3*$B5-D$3))-$C5
6	=IF(D$3>=3*$B6,$C$2*3*$B6,$C$2*D$3+$H$8*MIN($D$8,3*$B6-D$3))-$C6
7	0.2

The entire spreadsheet in numerical mode is:

	A	B	C	D	E	F	G	H	I
1			Price	Demand for Tickets					
2	Theatre	Number	$10	Fringe	Average	Great	Heavy	Expected	
3	Size	of Seats	Rent	250	800	2300	4500	Value	
4	Small	100	$600	2,000	2,400	2,400	2,400	2,320	
5	Medium	400	$1,800	1,300	6,800	10,200	10,200	6,040	Best
6	Large	1200	$4,700	-1,600	3,900	18,900	31,300	4,424	
7			Probability	0.20	0.70	0.09	0.01		
8	Salvage model with up to			300	last-minute tickets priced at			$2	

Note that a few words have been added on the spreadsheet to make it clear that we are looking at a variation of the basic model in which up to 300 last-minute tickets may be sold at a discount price of $2.00 each.

When a model has been made using a spreadsheet, it is easy to see what happens if one or more of the assumptions of the model is changed. For example, suppose that we wish to see what would happen if up to 750 (rather than just 300) discount-priced last-minute tickets could be sold. All we need do is replace the 300 in cell D4 with 750. Doing this, and then pressing the Enter key, we obtain:

	A	B	C	D	E	F	G	H	I
1			Price	Demand for Tickets					
2	Theatre	Number	$10	Fringe	Average	Great	Heavy	Expected	
3	Size	of Seats	Rent	250	800	2300	4500	Value	
4	Small	100	$600	2,000	2,400	2,400	2,400	2,320	
5	Medium	400	$1,800	2,200	7,000	10,200	10,200	6,360	Best
6	Large	1200	$4,700	-700	4,800	19,800	31,300	5,315	
7			Probability	0.20	0.70	0.09	0.01		
8	Salvage model with up to			750	last-minute tickets priced at			$2	

Even with this assumption, though the expected values for the medium and large theatres change, the recommendation for the theatre rental remains with a medium-sized theatre.

A decision analysis model will always be mathematically easy to solve, but whether or not we have solved the real problem (where to hold the play) depends heavily on the assumptions on which the model is based.

8.1.4 Expected Value of Perfect Information

Suppose that in situations of decision making under uncertainty, it might be possible to obtain perfect information about the uncertain event. For example, suppose that tomorrow's weather will be either sunny, cloudy, or rainy. Perfect information about this event would imply that today's forecast for tomorrow is certain to be correct. Of course, a perfect weather forecast is not possible, but the hypothetical construct of perfect information is useful because it establishes an upper bound for the expected value of any information about the event. For example, if a person would pay \$5.00 to hear a perfect weather forecast, then a real forecast can be worth no more than \$5.00.

We are interested in determining the *expected value of perfect information* (EVPI). We now show how to calculate the EVPI, using the theatre problem as an example. In this case, the uncertainty is the level of demand. Having perfect information means that we are told the demand level before having to commit to one of the three theatres. With perfect information we can choose the best alternative with respect to the level of demand. For any level of demand, we are interested in the highest payoff (i.e. the highest payoff in the *column*). We recall the payoff matrix for the basic model (i.e. no salvage value), and on this we highlight the best payoff in each column:

			Demand for Tickets				
Theatre	3-Night		Fringe	Average	Great	Heavy	Expected
Size	Capacity	Rent	250	800	2300	4500	Value
Small	300	\$600	1900	2400	2400	2400	\$2300
Medium	1200	\$1800	700	6200	10,200	10,200	\$5500
Large	3600	\$4700	−2200	3300	18,300	31,300	\$3830
		Probability	0.20	0.70	0.09	0.01	

If we are told that the demand will be "fringe", then we will choose a small theatre for a payoff of \$1900 (the highest payoff in the "fringe" column). If we are told that the demand will be "average", then we will choose a medium-sized theatre for a payoff of \$6200. If we are told that the demand will be "great" or "heavy", then we will choose a large theatre, with a payoff of \$18,300 for "great" and \$31,300 for "heavy".

There are now two ways to complete the calculation of the EVPI.

Direct Calculation of the EVPI

The new information only has value if it would change the recommendation that we had before. Before receiving the perfect information, we would have recommended renting a medium-sized theatre. If the perfect information is that demand will be "average", then we will still make the same recommendation. However, in the other three outcomes of demand, we will change the recommendation, thereby increasing the payoff over what it would have been. If the perfect information is that demand will be "fringe", then we would change the recommendation from medium to small, thereby increasing the payoff from 700 to 1900. If the perfect information is that demand will be "great", then we would change the recommendation from medium to large, thereby increasing the payoff from 10,200 to 18,300. If the perfect information is that demand will be "heavy", then we would change the recommendation from medium to large, thereby increasing the payoff from 10,200 to 31,300. The probabilities of the perfect information being that these outcomes will occur are 0.20 for "fringe", 0.09 for "great", and 0.01 for "heavy". Hence there is a 20% chance of increasing the payoff from 700 to 1900, a 70% chance of the payoff remaining at 6200, a 9% chance of increasing the payoff from 10,200 to 18,300, and a 1% chance of increasing the payoff from 10,200 to 31,300. The EVPI is therefore:

$$\begin{aligned}
\text{EVPI} &= 0.20(1900-700)+0.7(6200-6200)+0.09(18,300-10,200) \\
&\quad + 0.01(31,300-10,200) \\
&= 0.20(1200)+0+0.09(8100)+0.01(21,100) \\
&= 240+0+729+211 \\
&= 1180
\end{aligned}$$

The expected value of perfect information in the theatre example is \$1180.

Indirect Calculation of the EVPI

To indirectly calculate the EVPI, we first find the expected value *with* perfect information. To avoid confusion with the EVPI, the short form is EV with PI. The EV with PI is found by calculating the expected payoff based on the best alternative for each outcome. This is done by calculating the expected payoff using the highest payoff in each column.

$$\text{EV with PI} = 0.20(1900)+0.70(6200)+0.09(18,300)+0.01(31,300)$$

$$
\begin{aligned}
&= 380 + 4340 + 1647 + 313 \\
&= 6680
\end{aligned}
$$

The EV with PI is \$6680. If we did not have the perfect information, we would have chosen the medium-sized theatre alternative, which has an expected payoff of \$5500. The EVPI is the expected amount of the profit increase from not having perfect information to having it. The EVPI is therefore:

$$
\begin{aligned}
\text{EVPI} &= \text{EV with PI} - \text{EV without PI} \\
&= 6680 - 5500 \\
&= 1180
\end{aligned}
$$

As before, the EVPI is \$1180. You are expected to know how to calculate the EVPI using both of these methods.

For the theatre example, the \$1180 establishes an upper bound to the value of any information concerning the demand. If more information were available, the most that they would pay for it would be \$1180. If the price were say \$500, then it might be worthwhile purchasing it; it would depend on how good the information is. However, if the price were \$2000, it would not be worth purchasing no matter how good it is. While a small theatre company would not try to obtain more information about the demand, a company with millions of dollars at risk probably would.

8.1.5 Decision Criteria

Up to this point our sole decision criterion has been Expected Value. For a profit maximization example, we would choose the alternative with the highest expected value. For a cost minimization example (in which all the payoffs are costs) we would choose the alternative with the lowest expected value. This will remain our preferred decision criterion, but there are other criteria as well, and they are reviewed here. They are illustrated using the theatre example:

			Demand for Tickets			
Theatre Size	3-Night Capacity	Rent	Fringe 250	Average 800	Great 2300	Heavy 4500
Small	300	\$600	1900	2400	2400	2400
Medium	1200	\$1800	700	6200	10,200	10,200
Large	3600	\$4700	−2200	3300	18,300	31,300
		Probability	0.20	0.70	0.09	0.01

Pessimism

If a small theatre is chosen, the payoff will be either \$1900 or \$2400. Hence, the payoff will be at least \$1900. If a medium-sized theatre is chosen, the payoff will be either \$700, or \$6200, or \$10,200, hence the payoff will be at least \$700. From the four outcomes in the "Large" alternative row, we see that the payoff will be at least $-\$2200$. Of these three minimum payoffs, \$1900, \$700, and $-\$2200$, the highest is the \$1900 payoff. The alternative associated with this payoff is the small theatre.

Recommendation For a pessimist, renting the small theatre would be best.

For any maximization problem, the alternative associated with pessimism is the one which contains the maximum of the row minimums. [Note: In this example, all the row minimums were in the same column, but this will not be true in general.] For any *minimization* problem, the alternative associated with pessimism is the one which contains the minimum of the row maximums. Pessimism is an extreme form of risk-aversion which ignores all the information about probabilities.

Optimism

An optimist seeks the maximum for each alternative, and then seeks the maximum of the maximums. For the theatre example, the row maximums for Small, Medium, and Large are \$2400, \$10,200, and \$31,300 respectively. The maximum of these three is \$31,300, and the alternative associated with this payoff is the large theatre.

Recommendation For an optimist, renting the large theatre would be best.

Like pessimism, optimism ignores the information about probabilities. When optimism is applied to a cost minimization problem, we find the minimum of the row minimums.

Hurwicz

The Hurwicz criterion is a mixture of the criteria of Pessimism and Optimism. Either a coefficient of Pessimism (CoP) or a coefficient of Optimism (CoO) (one is the complement of the other) is chosen, and then (for maximization) a weighted average of the row minimums and maximums is found; the alternative with the highest weighted average is then chosen.

For the purposes of this course the CoP or CoO will be an exogenously given number. For example, suppose we wish to solve the theatre problem with an exogenously given coefficient of pessimism of 0.85. Hence, the coefficient of optimism is $1 - 0.85 = 0.15$, and we have:

	Pess.	Opt.	Hurwicz
Small	1900	2400	1975
Medium	700	10,200	2125
Large	−2200	31,300	2825
	0.85	0.15	

The highest weighted average in the Hurwicz column is 2825, which is in the large theatre row.

Recommendation Based on Hurwicz with CoP = 0.85, a large theatre would be recommended.

For a cost minimization problem, the pessimism column is based on row maximums, the optimism column is based on row minimums, and the chosen alternative is based on the lowest number in the Hurwicz column.

Laplace

The Laplace and Expected Value criteria are similar, except that for the Laplace equal probabilities are used. With n outcomes, the probability that any one of them occurs is $1/n$. The ranking for the Laplace criterion is conveniently found by summing, for every alternative, all n payoffs, and then dividing by n. We choose the highest ranking for maximization, and the lowest ranking for minimization.

For the theatre example with its four outcomes we have:

Small	(1900 + 3(2400))/4	=	2275
Medium	(700 + 6200 + 2(10,200))/4	=	6825
Large	(−2200 + 3300 + 18,300 + 31,300)/4	=	12,675

The highest of these is 12,675, which is associated with the large theatre.

Recommendation Based on the Laplace criterion the large theatre would be recommended.

The calculations for the four criteria of Pessimism, Optimism, Hurwicz, and Laplace can all be done on one payoff matrix. The best payoff for each criterion is highlighted; from these payoffs the best alternative for each criterion can be seen.

Theatre	Demand for Tickets							
Size	Fringe	Average	Great	Heavy	Pess.	Opt.	Hurwicz	Laplace
Small	1900	2400	2400	2400	1900	2400	1975	2275
Medium	700	6200	10,200	10,200	700	10,200	2125	6825
Large	−2200	3300	18,300	31,300	−2200	31,300	2825	12,675
					0.85	0.15		

The Regret Matrix

The *regret matrix* gives the cost of having *not* chosen, with hindsight, the best alternative for a given outcome. For example, in the theatre problem if the demand turns out to be "average", then the payoff is \$2400 for small, \$6200 for medium, and \$3300 for large. With hindsight, renting a medium-sized theatre is best for this particular outcome. In this case, there is no foregone profit. However, if the small theatre alternative were chosen, the foregone profit would be

$$\$6200 - \$2400 = \$3800$$

and if the large theatre alternative were chosen, the foregone profit would be

$$\$6200 - \$3300 = \$2900$$

The foregone profit is also called the *opportunity loss*. To find the opportunity loss for each situation from an already existing payoff matrix, we work with one column at a time. In every column of the payoff matrix, we subtract each number in the column from the largest number in that column. It is also possible to obtain the regret matrix without first finding the payoff matrix, and indeed one reason for obtaining the regret matrix is that it is sometimes easier to calculate than the payoff matrix. No matter how it is obtained, one property that the regret matrix will always have is that there must be at least one zero in every column.

In a payoff matrix we found the expected value for every alternative; in a regret matrix we find the *expected opportunity loss* (EOL) for every alternative.

It is found in an analogous manner to the EV – by finding the dot product of the probability row with every payoff row. The objective is to *minimize* the expected opportunity loss, so the best alternative is the one with the lowest number in the EOL column. The regret matrix along with EOL column for the theatre example is:

Regret Matrix			Demand for Tickets				
Theatre	3-Night		Fringe	Average	Great	Heavy	
Size	Capacity	Rent	250	800	2300	4500	EOL
Small	300	$600	0	3800	15,900	28,900	$4380
Medium	1200	$1800	1200	0	8,100	21,100	$1180
Large	3600	$4700	4100	2900	0	0	$2850
		Probability	0.20	0.70	0.09	0.01	

Recommendation Rent a medium-sized theatre, with an expected opportunity loss of $1180.

Some interesting comparisons can be made with the solution obtained by using a payoff matrix:

1. We obtained the same alternative using the regret matrix as we did when using the payoff matrix. This is not a coincidence – whether we maximize the expected value or minimize the expected opportunity loss, we always obtain the same alternative. Hence minimizing EOL, unlike pessimism, optimism, Hurwicz, and Laplace, is not a new decision criterion, but instead is just a variation on the maximizing expected value approach.

2. The minimum EOL, which is $1180, equals the EVPI. Again, this is not a coincidence – it is always true. While this gives us a third method for finding the EVPI, usually one does not take this approach to finding it if the payoff matrix has already been found.

3. For every alternative, the sum of the expected value and the expected opportunity loss is the same. Moreover, this sum is the EV with PI.

	EV	+	EOL	=	EV with PI
Small	2300	+	4380	=	6680
Medium	5500	+	1180	=	6680
Large	3830	+	2850	=	6680

This property is true for all examples.

8.1.6 Marginal Analysis

Some problems involving one decision and one event can be solved by a method that requires less work than is required for making a payoff matrix. This new method is called *marginal analysis*. It is applicable to problems in which there is a cost per unit ordered (w), a price at which items are sold (r) where the demand has any kind of discrete distribution ($P(d)$), and a price per unit at which all leftover items are sold (s). The marginal analysis method is not applicable for irregular problems such as the theatre example.

Let x (an integer) represent the optimal order quantity. The parameters of the model are:

Symbol	Meaning
r	retail price
s	salvage price
w	wholesale price

The context requires that the retail price be greater than the wholesale price, for otherwise the business could not exist. Also, the salvage value must be less than the wholesale price, for otherwise any amount of stock could be ordered at no risk to the retailer. Putting these observations into symbolic terms we have:

$$r > w > s$$

The distribution $P(d)$ gives the probability that the demand at price r is for *exactly* d units. We let $F(d)$ represent the *cumulative* probability function, which is the probability that d *or fewer* units are demanded. We can write $F(d)$ in terms of $P(d)$ as follows:

$$\begin{aligned}
F(0) &= P(0) \\
F(1) &= P(0) + P(1) \\
F(2) &= P(0) + P(1) + P(2) \\
F(3) &= P(0) + P(1) + P(2) + P(3)
\end{aligned}$$

and so on. Also, for $d \geq 1$, we can use the recursive formula:

$$F(d) = F(d-1) + P(d) \qquad (d \geq 1)$$

The optimal order quantity is given by the marginal analysis formula. It is not proved here, because the proof is somewhat advanced for an introductory course.

[You will not be tested on the proof – all that is required is that you know how to use the formula.] The value of x (the optimal order quantity) is chosen such that:

$$F(x-1) < \frac{r-w}{r-s} \leq F(x) \tag{8.2}$$

Another way of saying this is that we want the *smallest* value of x such that:

$$\frac{r-w}{r-s} \leq F(x)$$

An Example

Suppose that a vendor of digital pianos can buy pianos from the wholesaler for \$1325 each. The retail price per piano will be \$1600, but if any are left unsold by the end of the year, they will be priced to clear at \$1200 each. The retailer believes that at the \$1600 price at least five pianos can be sold, and as many as nine could be sold according to the following probability density function: five 10%, six 20%, seven 30%, eight 25%, and nine 15%.

Hence $r = 1600$, $w = 1325$, and $s = 1200$. Therefore,

$$\begin{aligned} \frac{r-w}{r-s} &= \frac{1600-1325}{1600-1200} \\ &= \frac{275}{400} \\ &= 0.6875 \end{aligned}$$

The demand is governed by $P(5) = 0.1$, $P(6) = 0.2$, $P(7) = 0.3$, $P(8) = 0.25$, and $P(9) = 0.15$. All $P(d)$ from $d = 0$ to $d = 4$ inclusive are 0, hence $F(d) = 0$ from $d = 0$ to $d = 4$ inclusive. Hence

$$\begin{aligned} F(5) &= F(4) + P(5) \\ &= 0 + 0.1 \\ &= 0.1 \end{aligned}$$

We can make a table to find $F(d)$, in which d goes from 5 to 9 inclusive, $P(d)$ comes from the given probabilities, and $F(d)$ is found recursively (for example the two numbers highlighted in blue are summed to find the number highlighted in red):

d	5	6	7	8	9
$P(d)$	0.1	0.2	0.3	0.25	0.15
$F(d)$	0.1	0.3	0.6	0.85	1.0

The critical value is at 0.6875; the number just above this in the $F(d)$ line is 0.85 (highlighted in yellow), and this is in the $d = 8$ column. In terms of the formula we have:

$$F(8-1) = 0.6 < 0.6875 \leq 0.85 = F(8)$$

Recommendation Order eight digital pianos.

Note that while the formula gives us the best order quantity, it does not also give us the expected payoff associated with this quantity. If we want this too, we can easily find it by solving for only the optimal row in the payoff matrix.

8.2 Decision Trees 1

8.2.1 Introduction

In the previous section, we examined simple situations in which there was only one decision, and each alternative was followed by the same event (same outcomes, with the same probabilities). Of course, problems in real-life are not that simple. Even when there is only one decision, the alternatives may be followed by different events. Also, there may be multiple decisions to be made. Either of these complications means that a payoff matrix cannot be used. Beginning with this section, we show how to handle more complex examples. Such problems will be analysed by first drawing what is called a *decision tree*. The drawing of the tree, along with the definitions of any symbols or abbreviated forms used on the tree, constitutes the formulation of the problem. The problem is then solved by performing a *rollback procedure* on the tree. Finally, the recommendation should be stated clearly.

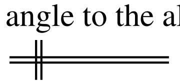
The overall procedure involves three phases:

1. The tree is drawn from left to right.
2. The rollback procedure is performed from right to left.
3. The recommendation is given in writing.

In this introduction we explain the graphical symbols used in the decision tree method. Just as real trees have branches, so do decision trees. On a decision tree, the point at which two or more branches meet is called a *node*. All decision trees have at least two kinds of nodes – decision nodes and event nodes. A decision node is drawn as a square, and an event node is drawn as a circle. Associated with each type of node is a corresponding branch emanating from the right side of the node. A decision node is followed by an alternative branch, and an event node is followed by an outcome branch. An alternative branch is represented by a double line, and an outcome branch by a single line.

Some textbooks only use what we have described so far, but we find that it is useful (whenever the decision criterion is expected value) to use three more symbols – a *cost gate*, a *payoff node*, and a *null branch*. A cost gate resembles a toll gate on a highway. It is drawn as two small squares (posts) joined by a straight line (the gate). The cost associated with the gate is written next to it; if it's a revenue rather than a cost, then the number is placed in parentheses. A payoff node is represented by a crossed circle, which is used to keep track of payoffs which occur before the ending branches of the tree. At the ending branch of the tree (on the extreme right-hand side), the payoff at that point is simply written down – a payoff node is not used. A payoff node is followed by a null branch, which is represented as a dashed line. Cost gates can appear on both alternative and null branches, usually representing a cost on the former, and a revenue on the latter. As an alternative to using cost gates (with the cost or revenue next to the gate), payoff nodes, and null branches, the maker of the decision tree could keep track of all costs and revenues and then subtract/add them to the appropriate final payoffs. Indeed, when dealing with utility functions (a topic which is beyond the scope of this book) that approach must be used. (This is why some textbooks avoid these three latter symbols.) However, whenever expected value is the decision criterion, using these three extra symbols makes the solution easier to find.

These symbols with their meanings are shown in Figure 8.1. These seven (or just four) symbols are all that are used when formulating a problem using a decision tree. However, when solving the tree (from right to left), we will use an eighth symbol, which is applied at every square to every branch except that of the best alternative at that square. This symbol consists of two short parallel lines which are drawn at a right angle to the alternative branch:

 non-recommended alternative

As has been stated, a decision tree is drawn from left to right, and is then solved ("rolled-back") from right to left. The tree is not drawn to scale with respect to

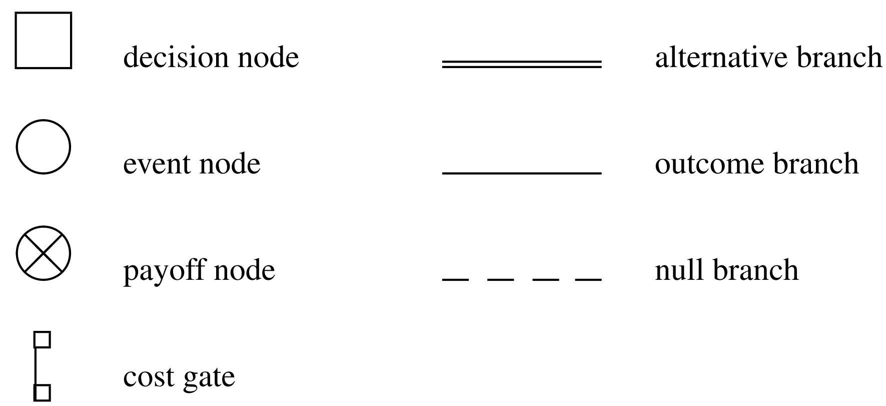
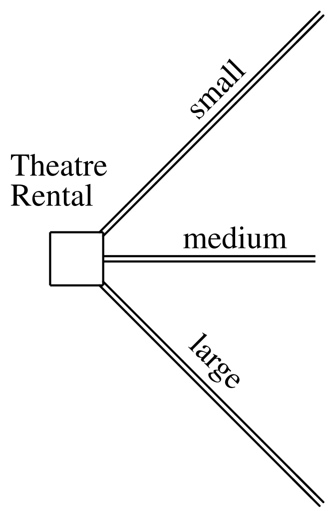


Figure 8.1: Symbols for Drawing Decision Trees (from Left to Right)

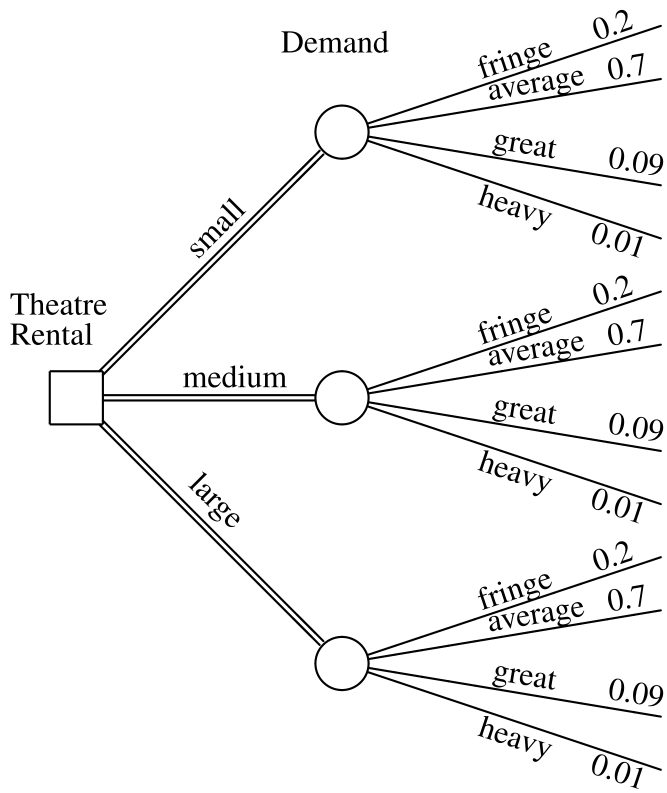
time, but the relative position in time must be preserved. Hence if something appears to the right of something else, then the thing on the right must come after (or be at exactly the same time) as the thing on the left.

8.2.2 Theatre Problem in Tree Form

To illustrate the nature of this approach, we will begin by formulating the theatre problem as a decision tree. [The problem description appears in Section 8.1.2.] This problem needs no payoff nodes; it can be done with or without cost gates on the alternative branches. We will do it both ways, first without cost gates to show the equivalence with the payoff matrix approach, and then with cost gates to show how these are used. The basic shape of the tree is the same for both approaches, so we will start with that. It should be emphasized that in using this method only one tree needs to be drawn. However, to illustrate this methodology, several trees are shown for one problem so that the order in which the material is drawn is made clear. We begin with a square on the left-hand side which represents the theatre rental decision. Emanating from the right-hand side of the square are three alternative branches (double lines). Next to these branches is a word describing the meaning of the alternative.

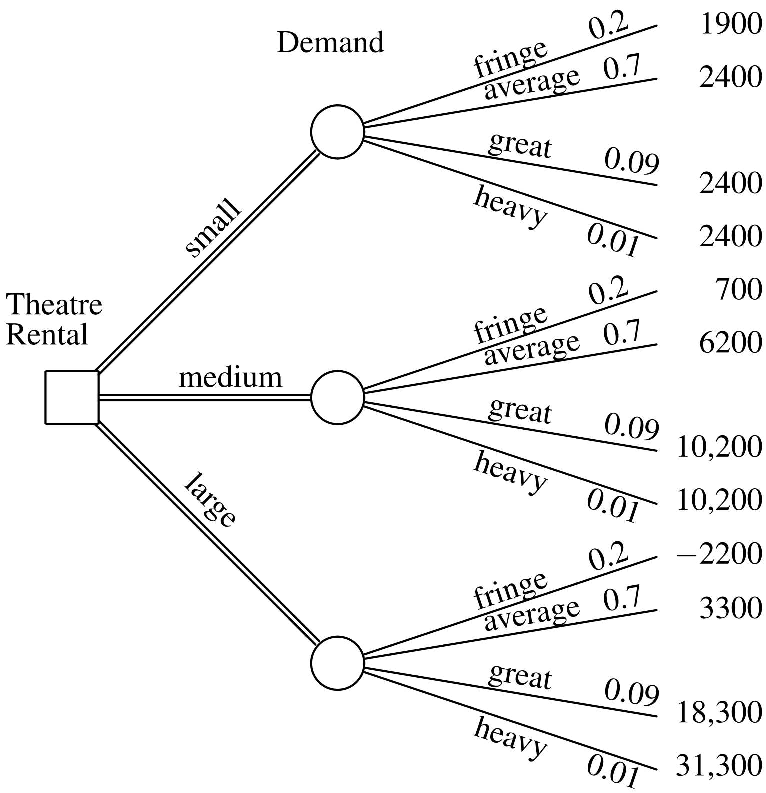


At the end of each of the alternative branches, there is a circle for the demand for tickets. Coming out of each circle there are four outcome branches (single lines). Again, words are written to describe the meaning of each outcome. Adding all this to the tree we obtain:



Without Cost Gates

When cost gates are not used, all costs are imbedded in the final payoffs. There are twelve final branches, and the payoffs which go to their right are in fact the twelve numbers which we calculated earlier and placed in the main body of the payoff matrix. Writing these numbers onto the tree we obtain:



We have finished the left-to-right formulation of the model, and we now proceed with the roll-back procedure, which proceeds from right to left.

At each circle, we compute the expected value. Just as we saw when we did it as a payoff matrix, the expected payoff at the circle which ends the "small" alternative branch is:

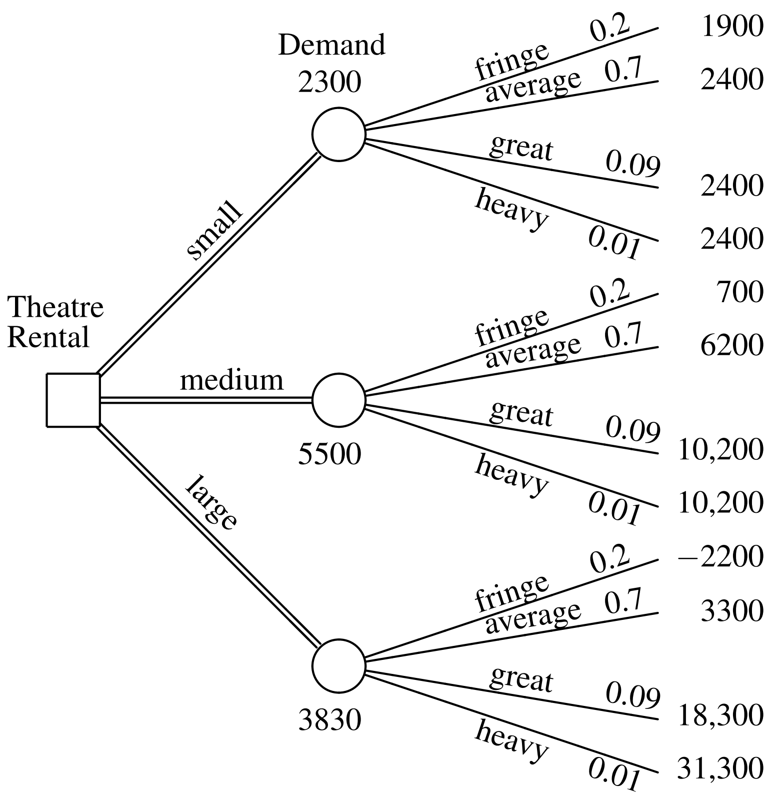
$$\begin{aligned}\text{EV(small)} &= 0.20(1900) + (0.70 + 0.09 + 0.01)(2400) \\ &= 380 + 1920 \\ &= 2300\end{aligned}$$

Similarly,

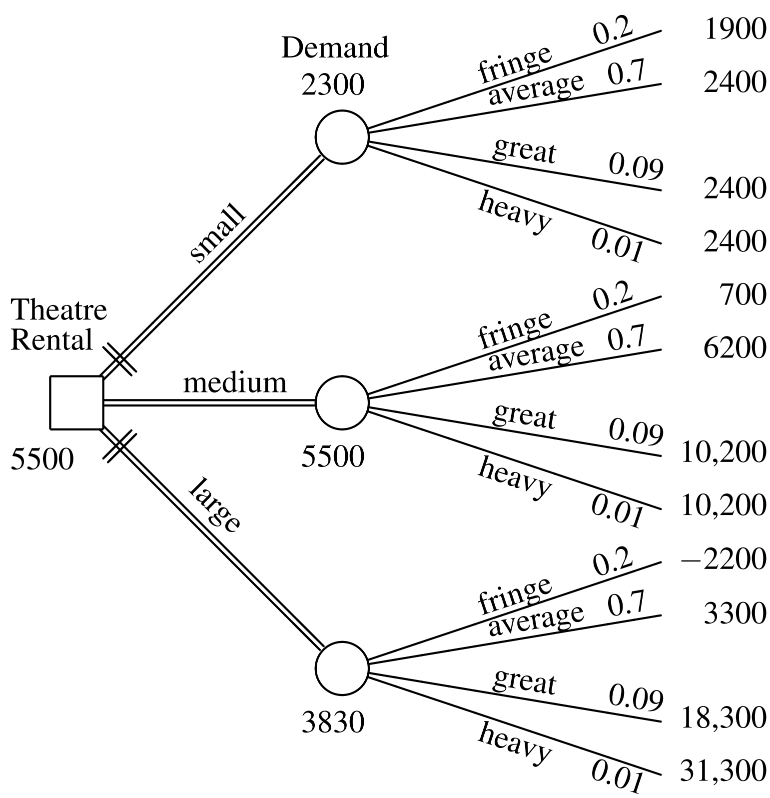
$$\begin{aligned}\text{EV(medium)} &= 0.20(700) + 0.70(6200) + (0.09 + 0.01)(10,200) \\ &= 140 + 4340 + 1020 \\ &= 5500\end{aligned}$$

$$
\begin{aligned}
\text{EV(large)} &= 0.20(-2200) + 0.70(3300) + 0.09(18,300) + 0.01(31,300) \\
&= -440 + 2310 + 1647 + 313 \\
&= 3830
\end{aligned}
$$

Putting these numbers from the rollback procedure onto the tree we obtain:



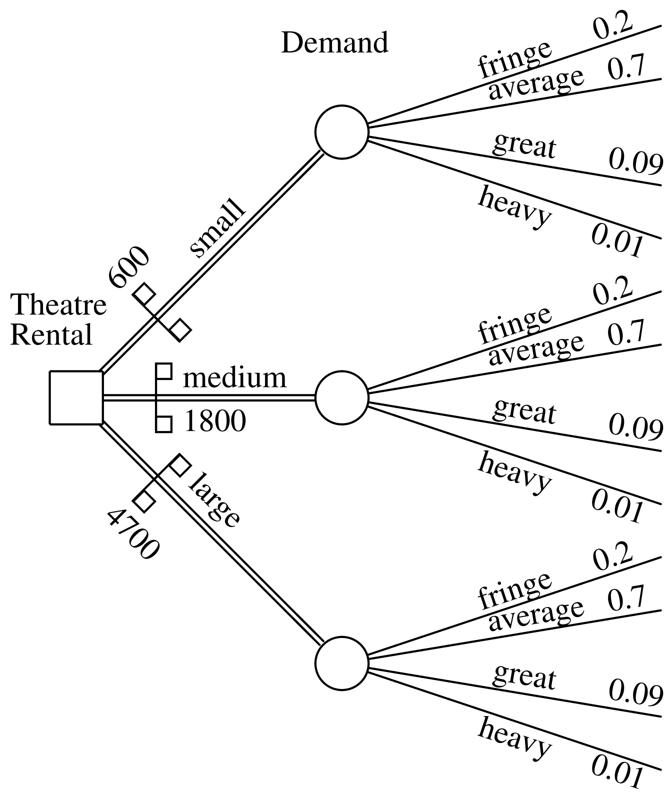
Moving to the left, we come to the square. At a square the best (highest, for profit maximization) payoff is chosen. Clearly, this is the $5500 associated with the medium-sized theatre alternative, and this number goes next to the square. The sub-optimal alternatives are marked with short double lines at right angles to the alternative branches. Putting these things on the tree we have:



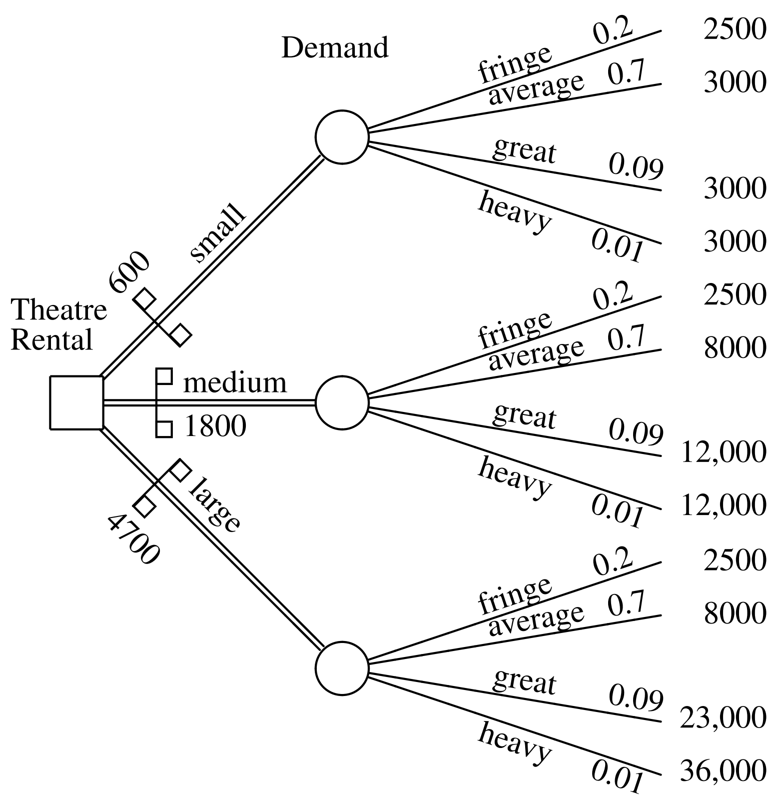
Recommendation Rent a medium-sized theatre, with an expected payoff of \$5500 before the deduction of \$2500 in fixed expenses, or \$3000 after making this deduction..

With Cost Gates

In the theatre problem there is a cost associated with each alternative, which is the rent for the theatre. Recall that this was \$600 for the small theatre, \$1800 for the medium-sized theatre, and \$4700 for the large one. Though we are starting with the tree having being drawn in this instance, normally one would begin to draw the tree and put on the cost gates as the alternative branches are drawn. The tree with cost gates (but without the final payoffs) is:



Now we must determine the final payoffs. For some problems, these payoffs are given exogenously in the problem description, but for this example we must work them out. These payoffs are all revenues from ticket sales. Recall that the small, medium, and large theatres can hold, over three nights, 300, 1200, and 3600 people respectively. The demand levels for fringe, average, great, and heavy are 250, 800, 2300, and 4500 respectively. The tickets net \$10 each. A small theatre obtains a revenue of \$10(250) = \$2500 with fringe demand, but otherwise the theatre is filled for a revenue of \$10(300) = \$3000. A medium-sized theatre has a revenue of \$2500 for fringe demand, \$10(800) = \$8000 for average demand, but otherwise the theatre is filled for a revenue of \$10(1200) = \$12,000. A large theatre has a revenue of \$2500 for fringe demand, \$8000 for average demand, \$10(2300) = \$23,000 for great demand, and is filled with heavy demand with a revenue of \$10(3600) = \$36,000. An advantage of using the cost-gate approach is that it creates a fair amount of repetition in the final payoffs. Adding these payoffs to the tree we obtain:



Now we calculate the expected value at each circle. Normally we would not write all the details out; we would simply calculate the numbers and then write them on the tree. However, since this is the introductory section for this material, the full workings are shown:

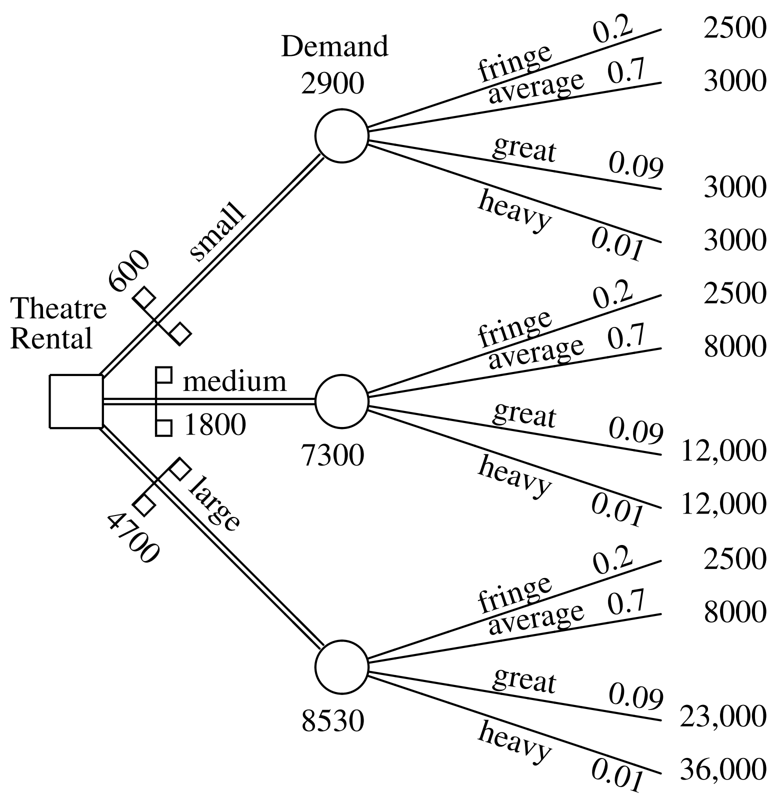
$$\begin{aligned}
\text{EV(small)} &= 0.20(2500)+(0.70+0.09+0.01)(3000)\\
&= 500+2400\\
&= 2900
\end{aligned}$$

$$\begin{aligned}
\text{EV(medium)} &= 0.20(2500)+0.70(8000)+(0.09+0.01)(12,000)\\
&= 500+5600+1200\\
&= 7300
\end{aligned}$$

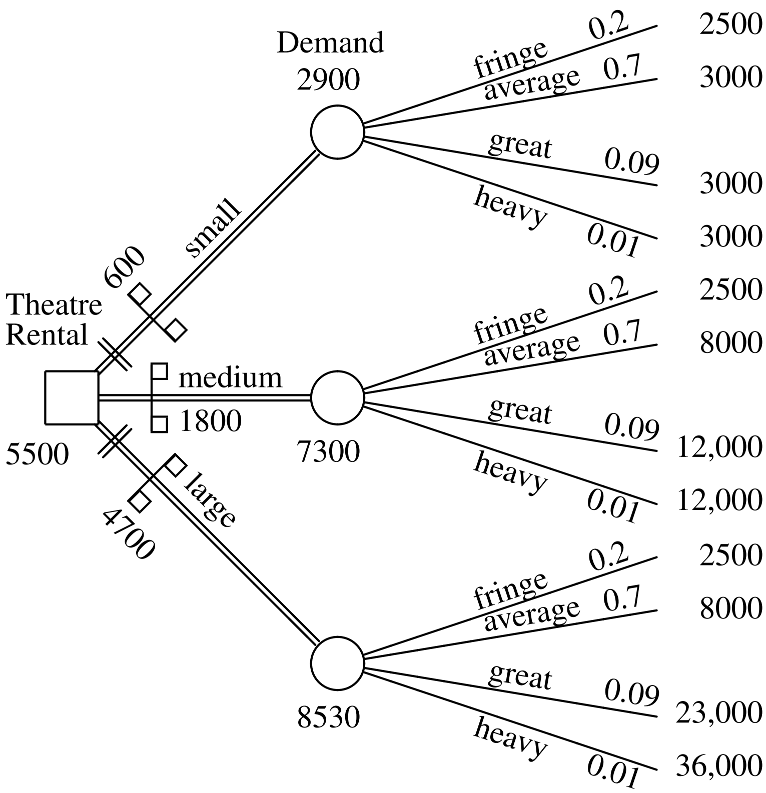
$$\text{EV(large)} = 0.20(2500)+0.70(8000)+0.09(23,000)+0.01(36,000)$$

$$\begin{aligned} &= 500 + 5600 + 2070 + 360 \\ &= 8530 \end{aligned}$$

Putting these numbers onto the tree we have:



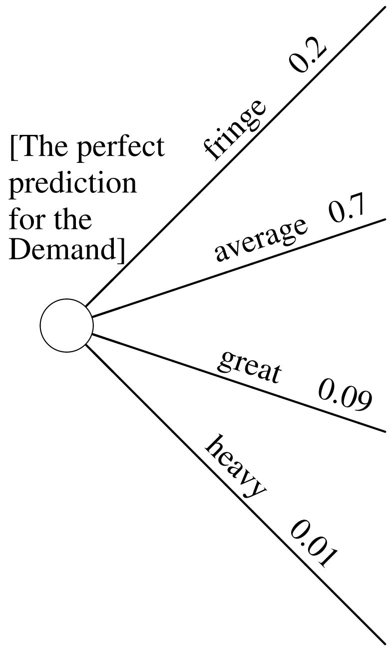
We are now interested in finding the highest net payoff at the square, each net payoff being the expected value at the circle minus the cost at the gate. The choices are: small, $2900 - 600 = 2300$; medium, $7300 - 1800 = 5500$; and large, $8530 - 4700 = 3830$. The best of these (as we saw before) is medium with an expected payoff of 5500. We write the 5500 next to the square, and draw short parallel lines through the sub-optimal alternatives to obtain:



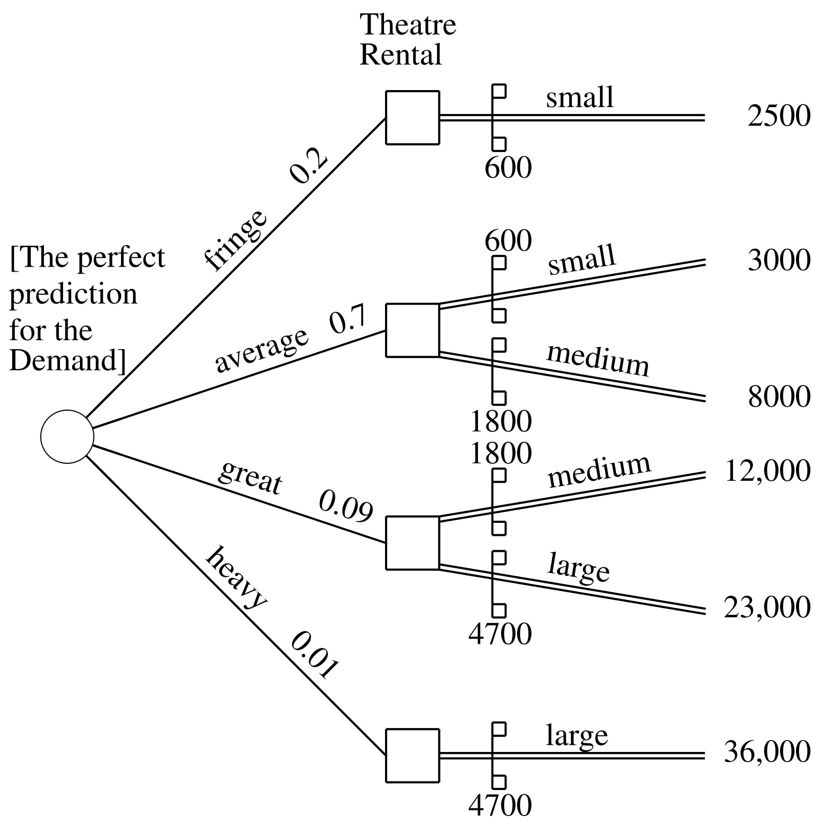
Of course, the recommendation is the same as before: Rent a medium-sized theatre, with an expected payoff of $5500 before the deduction of $2500 in fixed expenses, or $3000 after making this deduction..

8.2.3 The Expected Value of Perfect Information

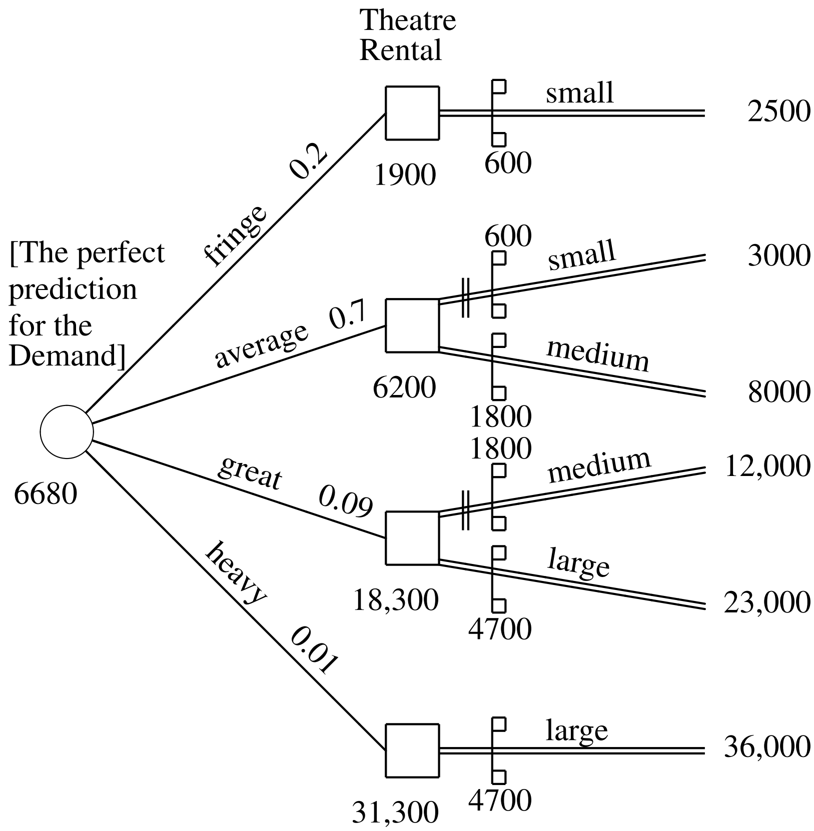
To find the EVPI using a decision tree, the event must precede the decision. This is because the decision maker receives the perfect information (which is that a particular outcome will occur) and then chooses the best alternative afterwards. Technically, the event is not the demand *per se*, but instead is the *prediction* about the demand. However, because the prediction is perfect, it has the same outcomes and probabilities as the demand itself. The tree therefore begins with an event node, followed by the four outcomes.



For each outcome, we can simplify the choices down to one or two reasonable alternatives. For example, if we are told that there will be fringe demand, it makes no sense to pay more rent for a medium-sized or large theatre, when a small one can easily handle all the demand. At the other extreme, not even the large theatre can handle heavy demand, so we wouldn't even consider a small or medium-sized theatre in this situation. With the other outcomes, it's not clear whether we should choose a theatre which is a size less than the demand (to save on the rent), or whether we should rent a theatre which is a size bigger than the demand. Hence with average demand we could investigate both a small and a medium-sized theatre, and with great demand we could consider both a medium-sized and a large theatre. Adding the reasonable alternatives with their cost gates and final payoffs, we obtain:



We now perform the rollback to obtain:



Calculating the EVPI by the indirect method, we obtain (as we did before):

$$\begin{aligned} \text{EVPI} &= \text{EV with PI} - \text{EV without PI} \\ &= 6680 - 5500 \\ &= 1180 \end{aligned}$$

The expected value of perfect information is \$1180.

8.2.4 Sequential Decision Making

Example

Bill operates a hardware store doing a reasonable amount of business for its size. There's a possibility of a smelter being built nearby, which will boost the town's population and his business if it goes ahead. Because of this, Bill wonders whether or not he should expand the store. The company which would operate the smelter has said that they will know one way or the other by late September, but that would be too late to look for a contractor to get the work done before the onset of winter. At the present time, there's about a 40% chance of the smelter going ahead.

Bill figures that relative to the profit that he would make anyway, the expansion would generate a profit margin of \$5,000,000 (net present value, excluding the capital costs of the expansion) if the smelter goes ahead, but only about \$1,600,000 if it does not. A contractor has quoted him a firm construction cost of \$2,900,000, provided that a contract is signed by July. If he does nothing before October, he could then make a deal for the expansion. If the smelter company has then said that they are indefinitely delaying the project, then the \$2,900,000 price is still available, but with a surcharge of \$150,000 for winter work. On the other hand, if the smelter company is going ahead with the project, then the construction cost will jump to a total of \$4,500,000, because everyone will be looking for construction work to be done.

Solution

Bill has two opportunities to expand his store. He could do it in July, when the construction cost would be lowest. Alternatively, he could do it in October, after he hears about a proposed smelter. The tree for this problem will have three parts: a decision about expanding now, an event concerning the smelter, and a decision about expanding in October.

We begin the tree by considering only the initial decision. He can either expand now, or wait to hear about the smelter. Expanding now would cost \$2,900,000, which we write on the tree as 2.9, making a note that all financial information is in millions of dollars.

The initial formulation of the tree is shown in Figure 8.2.

No matter which alternative is chosen, we then hear some information about the smelter. Either we find out that the smelter company is proceeding with its construction, or they have decided to delay construction for an indefinite period. While this is a decision from the point of view of the smelter company, it is an

Note: All financial information
is in millions of dollars.

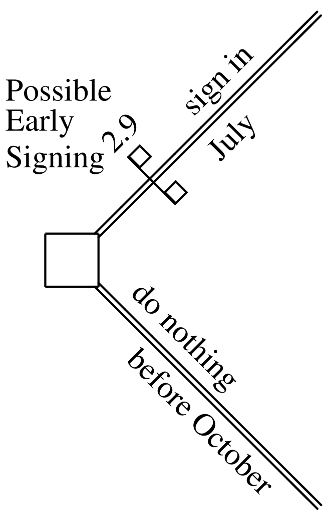


Figure 8.2: Bill's Hardware Store: Initial Part of the Formulation

event from Bill's perspective, because he has no control over it. Adding this event with its two outcomes, we obtain the partial formulation of the tree shown in Figure 8.3.

On the top part of the tree we place the two payoffs, which are the $5,000,000 and $1,600,000 figures mentioned in the text of the problem. On the bottom part of the tree, we draw the structure for the second decision. The construction costs are different from what they were before, either because of a price increase caused by all the smelter activity, or because of the extra cost for construction during the winter. There are four final payoffs; the two of these which are 0 are not mentioned explicitly in the problem description. For these we must see that if Bill does not expand his store, then the profit margin relative to what he is doing now must be 0. The complete formulation of the tree is shown in Figure 8.4.

Proceeding from right to left, we rollback the tree, finding the highest net payoff at each square, and the expected payoff at each circle. The rollbacked tree is shown in Figure 8.5.

The recommendation is to wait until October. If it's announced that the smelter will proceed, then Bill should expand his store. If it's announced that the smelter is indefinitely delayed, then Bill should do nothing. The ranking payoff is $200,000.

Finding the EVPI

We find the EV with PI by using a tree as shown in Figure 8.6. Note that because all the information comes at the outset, that if the expansion is to be done at all it would be best to do it in July, when the construction cost would be lowest.

The EV with PI calculated on the tree is 0.84, i.e. $840,000. Since we would have obtained $200,000 without the perfect information, the EVPI is:

$$\begin{aligned}
\text{EVPI} &= \text{EV with PI} - \text{EV without PI} \\
&= \$840,000 - \$200,000 \\
&= \$640,000
\end{aligned}$$

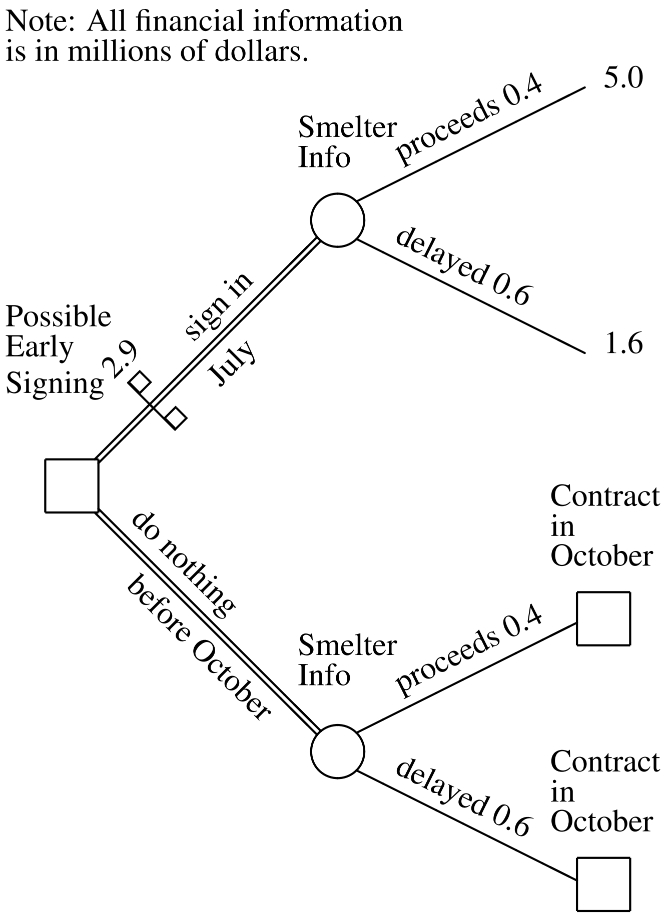


Figure 8.3: Bill's Hardware Store: Partial Formulation of the Tree

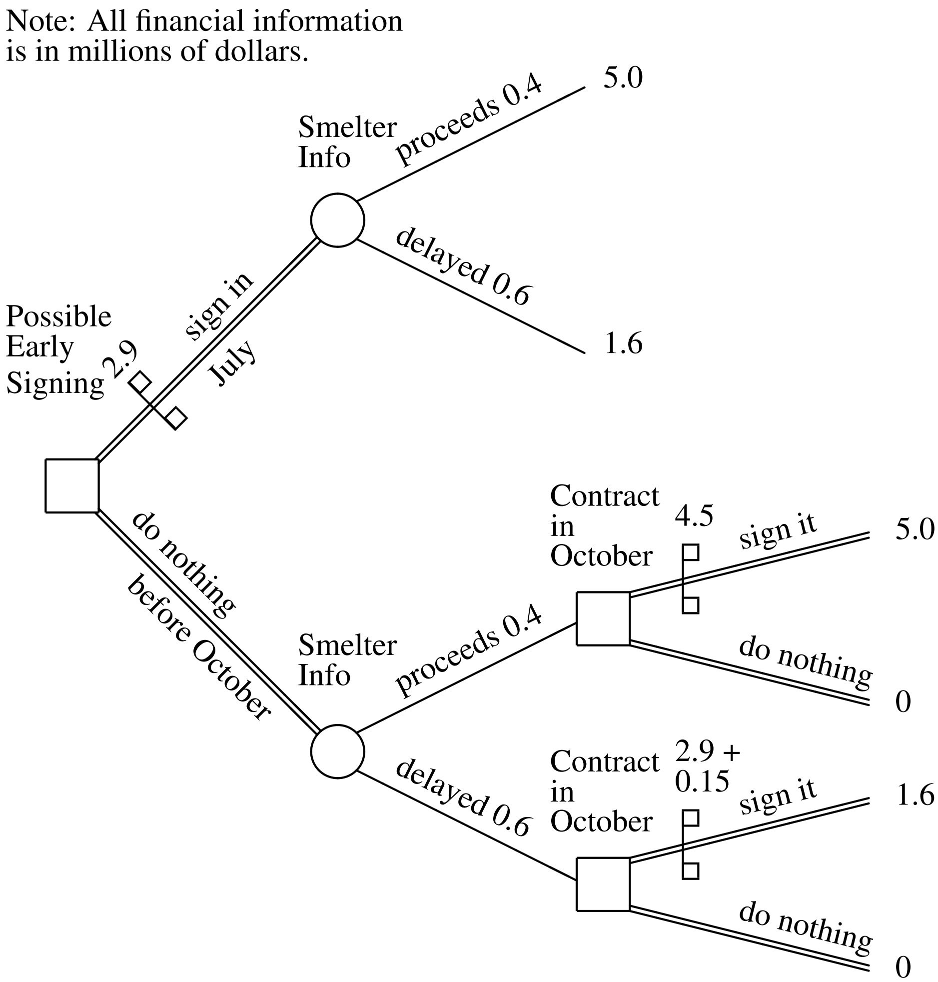


Figure 8.4: Bill's Hardware Store: Complete Formulation of the Tree

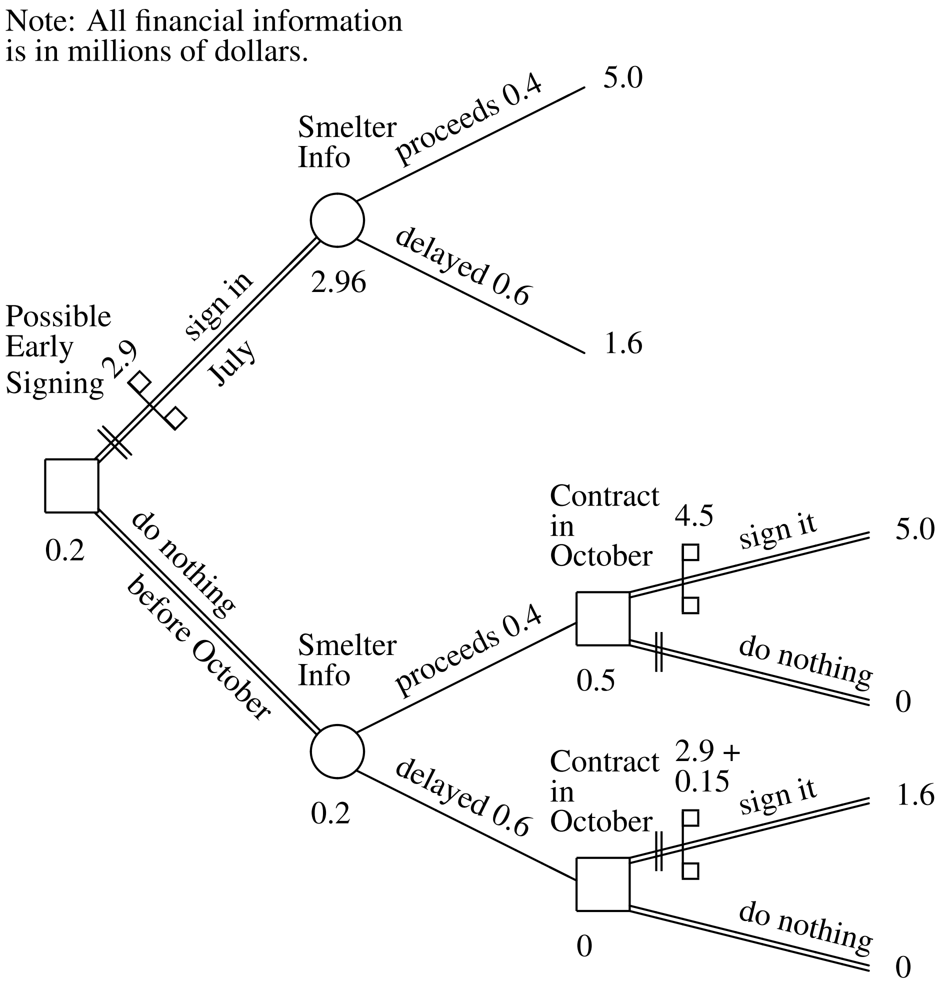


Figure 8.5: Bill's Hardware Store: Rollbacked Tree

Note: All financial information
is in millions of dollars.

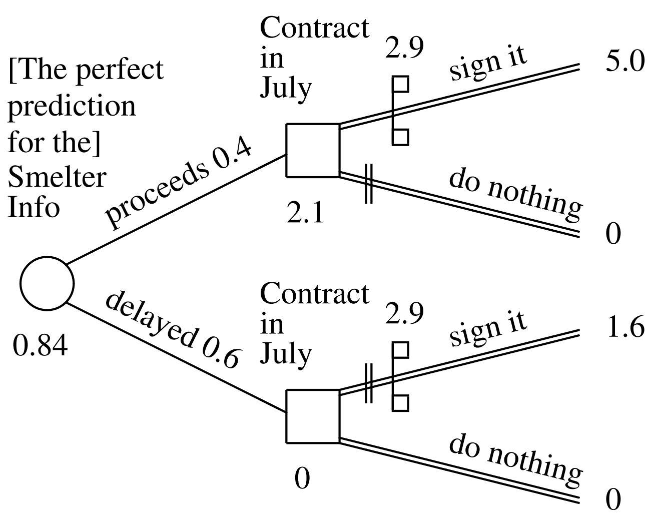


Figure 8.6: Bill's Hardware Store: Finding the EV with PI

8.3 Problems for Student Completion

8.3.1 Technical Exercise

A business must choose one of four alternatives. After the alternative has been chosen, an event occurs. This event has three mutually exclusive outcomes. The conditional payoffs are given in the following table.

	O_1	O_2	O_3	EV
A_1	-50	70	80	
A_2	-20	-15	90	
A_3	200	-20	-40	
A_4	60	-40	60	
Prob.	0.3	0.2	0.5	

(a) Determine the best alternative by finding the expected value associated with each alternative.

(b) Determine the EVPI by first finding the EV with PI.

(c) Find the regret matrix, and find the best alternative by computing the EOL for each alternative.

8.3.2 Choosing a Concert Venue

A concert promoter is planning a single performance of J.S. Bach's *Mass in B minor* by a choir and organist. Several venues are being considered, all of which have a pipe organ, and are accessible for those with physical challenges. The seating capacities and rental fees are:

Venue	Seats	Rent
Church	500	$400
Cathedral	900	$800
Concert Hall	1400	$1100
Basilica	1500	$1200

All tickets sell for $30, but $5 of this is a ticket agency fee and taxes. In addition to the rents mentioned above, for all tickets sold beyond the first 800, the promoter will pay 20% of the incremental revenue to the venue. Hence the promotor receives a net of \$25 per ticket for the first 800 tickets, and 0.8(25) = \$20 per ticket for all other tickets sold. The promotor believes that the demand will be as follows: 400 30%; 700 25%; 1000 20%; 1300 15%; and 1600 10%. Aside from the aforementioned costs, there will be \$16,000 required for professional fees, advertising, and so on.

By hand, make a payoff matrix to determine which venue the promoter should rent.

8.3.3 Computer Retailing

A computer retailer is about to order some computers from the manufacturer. Over the next two months the retailer believes that there is a 20% chance of demand for ten computers, a 30% chance of demand for eleven, a 20% chance of demand for twelve, a 20% chance for thirteen, and finally a 10% chance that fourteen computers will be demanded. On a per-unit basis, the wholesale cost is \$1500, and the retail price is \$1950. Any computers leftover after the two month period can be sold without problem at \$1400 each.

(a) Solve this problem by hand by using a payoff matrix. Rather than develop a formula, it is easier to first find out what happens if 10 computers are ordered, and then 10 are demanded. Then, calculate the payoff when 11 are ordered, and 11 are demanded, and then continue to find the rest of the payoffs on the main diagonal. Next, write the payoffs in the top right-hand triangle (very easy). Finally, write the payoffs in the bottom left-hand triangle. Begin this latter step by determining what happens if 11 computers are ordered, but only 10 are demanded. By how much worse should this be than the (10,10) situation? By how much worse should this be than the (11,11) situation? Both answers should lead to the same payoff for the (11,10) case. Continue in this manner to find all the payoffs. Then, compute the expected values and determine the best policy.

(b) Determine a recommendation by using a spreadsheet.

(c) Determine a recommendation using the marginal analysis formula. [Obviously, (a), (b), and (c) should produce the same result.]

(d) Determine the EVPI both directly and indirectly.

(e) Which alternative would be picked using: (i) Pessimism; (ii) Optimism; (iii) Hurwicz with a coefficient of optimism of 0.8; (iv) Laplace?

(f) By hand, find the regret matrix, and determine the alternative with the minimum EOL. Try to find the regret matrix just from the data of the problem, without looking at the payoff matrix from part (a).

8.3.4 Marginal Analysis Problem

A vendor has found that demand for newspapers can vary from 31 to 70 inclusive with each number being equally likely. Newspapers are bought by the vendor at 50 cents each, and are sold for 75 cents each. Any left-over copies at the end of the day are sold to a recycling operation at 5 cents per copy. By using the marginal analysis formula, determine the number of copies that the vendor should order.

8.3.5 Niagara Frontier Winery

Make a decision tree for the situation described in the following problem, and provide a recommendation.

Driving to her office in St. Catherines, Ontario, Betty Johnson, the production manager of *Niagara Frontier Winery*, heard a very disturbing weather forecast. "Environment Canada has issued a severe frost warning for the Niagara Region for later in the week". This was only the 27th of September, and the grapes would not be ready for harvest for another three weeks. Telephoning the weather office for more detailed information, she was told that in three days time there would be a 60% chance of a mild frost, and a 30% chance of a severe frost. A mild frost could at least be contained by erecting heaters in the fields at a cost of $350,000. Using heaters, the damage would be minimal; about 80% of the crop could still be made into high-quality wine, and a further 10% could be made into low-quality wine. A severe frost, on the other hand, would destroy the crop entirely; even an attempt at using heaters would be in vain. As long as she made up her mind by 1 p.m., there was enough time to erect the heaters. Also, there was enough time to order that all the crop be picked immediately, which would cost $400,000. It could either be sold as grape juice or made into low-quality wine at a value of $1,200,000. If the crop were picked in good condition in three weeks time, however, it would be worth about $3,000,000, but from this the $400,000 harvesting cost would have to be paid.

There was one more complication. The research department had come up with something called "ice-wine". If the heaters were not used and a mild frost was experienced, none of the crop could be made into high-quality wine, but perhaps this could be made into ice-wine. This would cost \$400,000 to pick the crop, and would have a 75% chance of success. A successful product would be worth \$2,600,000, but a failure would be worth nothing. Alternatively, about 85% of the crop with mild frost damage could be sold as low-quality wine. Betty made herself a pot of tea and then looked at her watch. The one o'clock deadline was fast approaching.

8.3.6 Ski Resort Snow System

A ski resort has always relied on natural snow, which could come in any one of four levels: heavy, medium, light, or none, with probabilities 0.1, 0.4, 0.3, and 0.2 respectively. They are now, in July, considering the installation of an artificial snow-making system before the upcoming season. If installed, the annual amortized cost would be \$40,000. The operating costs of the snow system would be \$0 if the natural snow were heavy, \$50,000 if medium, \$80,000 if light, and \$110,000 if there were no natural snow. With an artificial snow system, or with heavy natural snow, they would obtain revenue of \$200,000. With no artificial snow, the revenue would be only \$130,000 with medium snow, \$70,000 with little snow, and \$0 if there were no snow. Operating costs other than snow-making would be \$45,000 per year (whether artificial or natural snow).

Choosing, before the season begins, to close the operation completely for the upcoming season, would allow them to rent the land with a rental income of \$20,000.

(a) Solve this problem using a decision tree.

(b) Determine the EVPI by first finding the EV with PI.

8.3.7 Retailing Compact Discs

Despite the appeal of digital formats to some people, a record shop in the downtown area still sells used vinyl LPs and new compact discs. The owner of the store must decide which discs and the number of each to order for the Christmas sales season. A new compilation of classical music featuring Mozart's *Laudate Dominum*, the *Allegro* from Symphony No. 1 in B-flat Major by William Boyce, *Les Barricades Mystérieuses* by François Couperin, *Sarabande* by Handel, and many others, is sweeping the world. Orders for the disc must be placed with the distributor in lots of 100. If she orders 100 discs, the cost to her would be \$14 per disc; 200 discs would cost \$12 per disc, and 300 or more in lots of 100 would cost \$10 per disc. Until Christmas Day the retail selling price will be \$20 per disc; any left over after Christmas will be sold to a discount house in another city for \$5 per disc. The owner believes that at the regular price the possible demands are 50, 100, 150, 200, 250, 300, or 350 discs, with probabilities 0.05, 0.1, 0.2, 0.3, 0.2, 0.1, and 0.05 respectively. She must place her entire order now. Assume that she will suffer no loss of goodwill if she happens to be out of stock.

(a) Make and solve a model in Excel to provide a recommendation to the owner based on maximizing the expected profit.

(b) Determine the expected value of perfect information.

(c) Suppose that the \$5 to be received for each leftover disc is negotiable within the range \$0 to \$10. Over what range for this value would the recommended order quantity found in part (a) be valid?

   (i) This can be found by manually varying the number in whatever cell was used for the salvage value in part (a).

   (ii) This can also be done by using Goal Seek, which is found under Data/What-if Analysis. In Goal Seek there are three boxes to be filled in, called *Set cell:*, *To value:*, and *By changing cell:*. Make two cells which calculate the difference in Expected Profit between the optimal order quantity row and (i) the row above it, and (ii) the row below it. Now run Goal Seek twice, where the objective in each is to make one of these two cells equal to 0. Here, the cell which computes the difference is the *Set cell:*, the *To value:* is 0, and the *By changing cell:* is the cell which contains the salvage value.

Chapter 9

Decision Analysis II

9.1 Decision Tree with Payoff Nodes

In the previous chapter, we saw most of the technical operations to handle decision trees. If there's any difficulty using trees, it is the formulation – the rollback procedure is very straightforward. In this section we solve a fairly long case. Doing this case adds one more technical operation – the use of payoff nodes. More importantly, though, solving this case illustrates the application of the decision tree methodology to a somewhat complex situation.

9.1.1 Case: New Detergent Marketing Campaign

Elizabeth, John, and Susan work for a consumer products company. They come from widely different academic backgrounds. Before joining the company, Elizabeth obtained a B.Sc. and M.Sc. in biochemistry, and is now part of a research team which has come up with a new type of detergent. John earned a joint B.A. in English literature and art, worked for a while for a competitor, and now works with his present employer on all advertising campaigns. Susan obtained a B.Comm., specializing in marketing, but she was good at all things, including the course on decision modeling. She is now doing an MBA part-time, while working full-time.

They recently held a meeting to discuss what to do about the newly developed detergent. The meeting began with Elizabeth welcoming the others. "John and Susan, thanks for coming. The research team is very pleased with this new product. We tested it extensively in the laboratory, and found that there was virtually no fading of colours even after 100 washes. I hope that with your help we can bring this product to market." "John and I have read the report," Susan replied, "but it's the part where it says that the cost will have to be 20% higher than even full-priced brands that has me worried. I fear that when it comes to the typical shopper looking at the prices in the store, that a claim of technical excellence is not going to amount to much." "I've been thinking about that," said John. "We have to make it clear in the ad campaign that the consumer would be paying more for the detergent but would be saving much more than that in the long-term on the cost of replacing clothes. I grant you that the average shopper will be skeptical, but we hope that at least some segment of the market will understand the trade-off and therefore buy our product."

Susan knew that Elizabeth was excited about the new detergent because she had helped develop it, and that John was looking forward to a challenge in writing the ad copy. However, she also knew that only about one in ten new products eventually succeeded in the market place. Thinking that the others would want to proceed, she had come up with some approximate numbers. "I'm assuming that for now at least, our market is Canada," Susan said. "The United States is just next door, with nearly ten times as many people, but we don't have a distribution network there, so the best that we could hope for in the States is a licensing agreement several years down the road, if everything works out here first. For now, we should see if this product will be profitable in the Canadian market alone." Elizabeth and John nodded their heads, and Susan continued. "If we try for the whole Canadian market, the start-up costs would be about $800,000. After that would come some revenue, whether the product turns out to be a success or not. A success would bring in about $4,000,000, but a failure would provide only a tenth of that. If success or failure were 50/50, I'd proceed, but the chance of success is only one in ten."

Elizabeth wondered how accurate Susan's figures were. Perhaps if the start-up costs could be lowered, or the revenues raised, or the probability of success raised, the project would make sense. "Susan, your numbers are at best estimates. With different numbers this project could go ahead." "Sure," replied Susan, "and with different numbers the project could be even less viable than it is now. I'm not saying that this new detergent couldn't do well for us, but maybe we should try to test-market this product before launching it into the entire Canadian market." John broke in when he heard this idea. "We did test-marketing when I was at my former employer. Usually, if a product succeeded in the test-market, it did well everywhere. There were exceptions, though. Chocolate-covered seaweed did well when we tried it in Halifax, but bombed when we tried to go national – you couldn't give it away in Toronto. On the other hand, we test-marketed a new quick-cook rice in Regina, and it didn't do well, but when on a hunch we went ahead with a national campaign anyway, it suddenly became a success. It did best in cities with large immigrant populations, and in hindsight we saw that Regina wasn't a good test-market for that kind of product."

"You've hit on a good point, John," said Susan, "the test-market should ideally reflect the country as a whole, but that's not always easy to do. Since advertising is expensive, we concentrate on small geographic areas away from high-priced media buys in large cities. For example, Pickering [just east of Toronto] would be an expensive place to test-market, because we'd have to buy airtime on Toronto stations and pay to reach eight million people in central southern Ontario, when we only want to reach the ones who live in Pickering. In Ontario, Peterborough is often used as a test-market because we can buy airtime just in Peterborough at a reasonable price. For the same reason, test-marketing in Alberta is often done in Lethbridge, which is large enough to have its own media outlets, but doesn't have the high rates that are found in Edmonton and Calgary."

Elizabeth wondered aloud about some of John's comments. "What does it prove once we get the result from the test market? The detergent could be like the chocolate-covered seaweed, or like the rice, rather than being a perfect predictor for what should be done." "You're right, Elizabeth," replied Susan, "test-marketing is not a perfect predictor, but it should give us a better idea of what to do. If we believe that there's one chance in ten of the product being a success in the country, then there should be more-or-less a 10% chance of success in any test-market. However, we know from past experience that people in British Columbia are most open to new products, and this figure generally declines as one heads east. If we just test in one market, I'd say that there's about a 12% chance of success in Lethbridge, about 10% chance of success in Peterborough, and just 8% in St. John's. If we test in one of these places and it's a failure, then the chance of success in the rest of Canada certainly becomes less than 10% – I don't know how much less, but it really doesn't matter. This project is tenuous enough as it is, without having to deal with a negative test result. On the other hand, a success in a test market would be a good omen for the rest of the country. As John said, there's no guarantee of success elsewhere, but I have to believe that on average the chance of success has increased from 10% to say 60%, though I think that this figure would range from 50% in Lethbridge to 70% in St. John's. While the purpose of the test-marketing is to obtain information, there would be some revenues as well, perhaps $30,000 for a success, but only a tenth of that for a failure."

"We could test in two of these markets, or perhaps even all three of them," Elizabeth suggested. "But if we test in two markets our advertising costs would double, and if we test in three these costs will triple," John said. "Testing in all three is probably not going to fly," Susan said, "but a case could be made for testing in two markets. I think that Lethbridge and St. John's would give us a better sense of the country as a whole than using either of these cities with Peterborough. That brings us to the question of how we should use these two markets. Should we test simultaneously in both, or should we test sequentially, beginning with one of the two cities, and then based on what we find there, possibly proceeding to the other?"

At this point Elizabeth jumped in. "Whatever we make in the lab, someone else can make too. I worry that if we do the test marketing, a competitor will buy a litre of it, have it chemically analyzed, and then reverse-engineer it in their own labs. This would take some time, of course, but if we test in two markets, and do it sequentially rather than simultaneously, we might just give them the time that they need. After taking all the risk, we would then have to share the market with someone else." "Point taken," said Susan. "If we test in both Lethbridge and St. John's, let's agree that we will do the testing simultaneously. This gives us four possibilities for the test results. If we fail in both places, we can forget about proceeding further with this product. Should we succeed in both, I'm almost certain that we would have a winner on our hands; I'd put the probability at 0.99. If we fail in one, but succeed in the other, I would want to boost the advertising expenditures by \$50,000, and based on that I'd put our chances of success in the rest of the country at about 35%."

From his experience, John had some figures on test-marketing. "Before we spend anything on advertising, we would have to spend about \$15,000 to develop an ad campaign. The three test markets aren't much different in size. I'd say that in each the cost to buy air time would be about \$10,000." Elizabeth wondered if spending this money now would save some money later should they decide to undertake a national campaign. "Here's a hypothetical one for you, John. Suppose that we test in Lethbridge and St. John's, and both tests turn out to be a success, so we decide to go national. Having spent \$15,000 plus two times \$10,000 for a total of \$35,000, can we deduct this amount from the \$800,000 cost of the national campaign?" "I wouldn't count on that," John replied. "We would probably want to modify the test-market advertising, so that will cost money. More importantly, when we buy national advertising, we obtain economies of scale by making one nation-wide media buy. It would be cheaper to do it that way than to buy air time in every city individually except where we test-marketed. I'd say that no matter what we do with test-marketing, the cost of the national campaign would be \$800,000. At the same time, this would be a new campaign as far as the test-market is concerned, so the national revenues wouldn't be diminished."

"It's time to wrap this up for this morning," Susan said. "Senior management will want to see a business plan, and the basis for this will be the recommendation which will come from making a decision tree of what we've been discussing. I'll work on this later this morning, and we'll meet again at 2 p.m. to discuss it."

9.1.2 The EVPI

There are four pieces of uncertainty in the case: the result in each of the three test markets; and the result of a national campaign. While it would be possible to compute the EVPI based on knowing perfect information about any of these four things, or any combination of these four things, it is the uncertainty about the national campaign which is of primary importance.

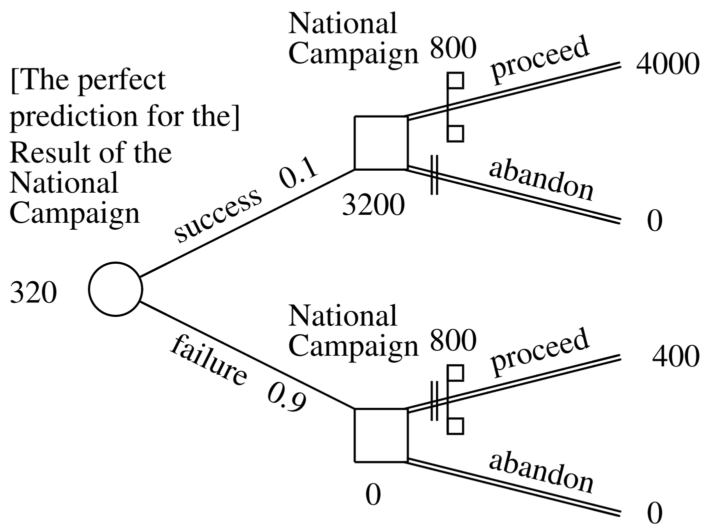
We can find the EVPI for a decision about the national campaign in the absence of test-marketing. If we know that the product would be successful, then clearly we would spend \$800,000 to make \$4,000,000, for a net of \$3,200,000. If we know that the product would be a failure, then clearly we would not spend \$800,000 to make only one-tenth of \$4,000,000. i.e. \$400,000. There is a 10% chance of being told that a success will occur, and a 90% chance of being told that a failure will occur, hence the EV with PI is:

$$0.10(\$3,200,000) + 0.90(0) = \$320,000$$

Without perfect information, and without test-marketing, we would not spend \$800,000 to obtain a 10% chance of making \$4,000,000. Instead, we would abandon this project, with the payoff without perfect information being \$0. Hence the EVPI is:

$$\begin{aligned} \text{EVPI} &= \text{EV with PI} - \text{EV without PI} \\ &= \$320,000 - \$0 \\ &= \$320,000 \end{aligned}$$

Though we did not need to draw a tree to find the EV with PI, we can do so if we wish. We begin with the event, being the prediction about the success or failure of the national campaign, followed by the decision about whether or not to proceed with the national campaign. Making this tree with payoffs in thousands of dollars and performing the rollback we have:



Hence the EV with PI is \$320,000, and then subtracting the EV without PI, which is \$0, we see that the EVPI is \$320,000. The cost of test-marketing is less than this figure, so we cannot rule out the possibility of using the test-marketing (which we would have done if these costs had exceeded the EVPI). Hence we need to continue with the analysis of this situation.

9.1.3 Formulation

We wish to develop and solve the decision tree to which Susan refers. It's too complicated to think of all the decisions and events at once in a long case like this. Instead, we should think about what must come first. It is more-or-less obvious that the case presents us with at least four alternatives for the test-marketing: Lethbridge only; Peterborough only; St. John's only; and testing simultaneously in both Lethbridge and St. John's. The three persons seem to agree that other types of multiple testing (sequential testing, or all three cities, or another pair of cities) should not be considered, and we will therefore leave these options out of the decision tree. At the other extreme, going directly to a national campaign without doing any test-marketing was not clearly opposed by Elizabeth, so we might wish to investigate this course of action. Also, we should consider doing nothing whatsoever, which for many business situations may be best of all.

Based on the foregoing, we could begin with a square followed by six alternative branches: one for each of the four testing alternatives; one for proceeding to national marketing directly; and finally a do-nothing alternative. Doing it this way

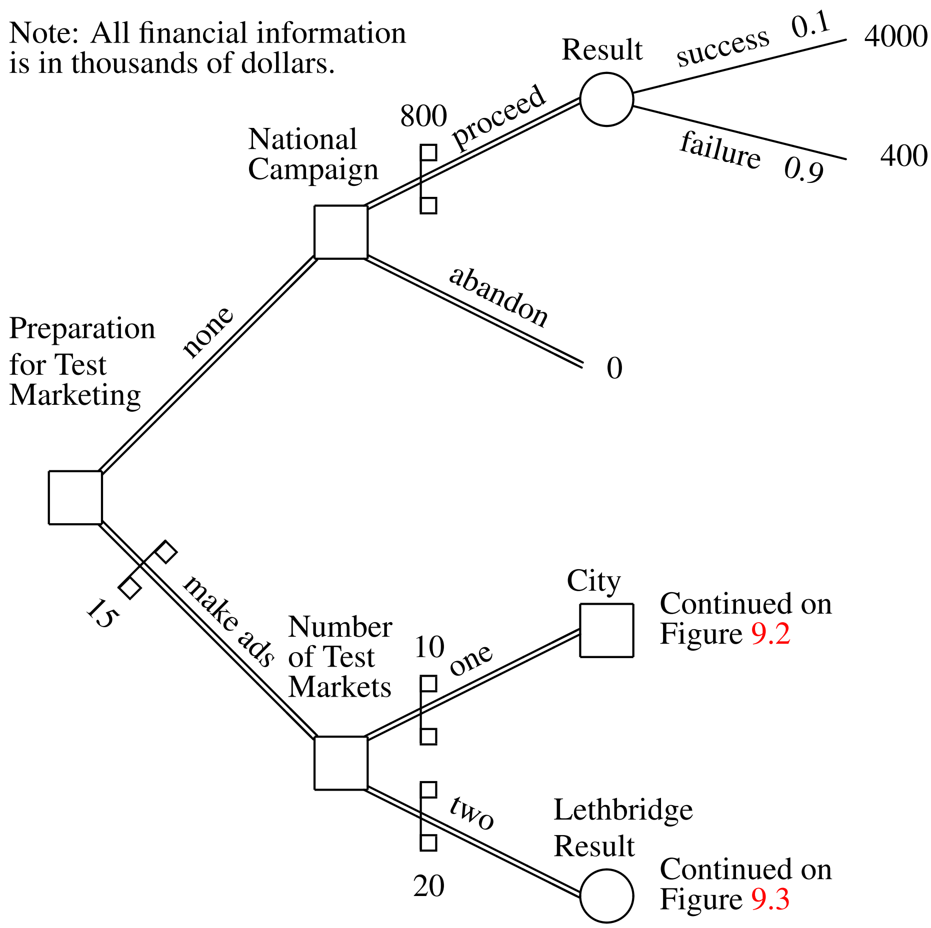
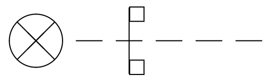


Figure 9.1: New Detergent Case: Beginning of the Tree

would be correct, but things become clearer if we first have a test-marketing decision which has just two alternatives: test market; and do not test market. The first of these alternatives then requires a decision about the manner of the test marketing. The second has a decision about the national campaign with two alternatives: proceed with the national campaign; or do nothing. Aside from the clarity provided by this approach, it allows the $15,000 cost of preparing the test-market ad campaign to be by itself on the test market alternative branch, with the advertising costs being handled separately. If we proceed with the national campaign with no test marketing, then this alternative is followed by a result event, with its two outcomes: there is a 10% chance of making $4,000,000, and a 90% chance of making $400,000. Because of space limitations all financial figures will be written on the tree in thousands of dollars, hence for example $4,000,000 is written on the tree simply as 4000.

The manner of the test marketing could be one decision with four alternatives, but again it makes things conceptually easier if we have two decisions. First, we decide whether we want one or two test markets. If one test market is chosen, then we must decide whether it will be in Lethbridge, Peterborough, or St. John's, and if we want two test markets it is understood from the case that these will be in Lethbridge and St. John's, hence the event for the result on one of the test markets comes next. In making the tree it turns out that we already have too much to put on one piece of letter-size paper. Hence, on this piece of paper we end with two nodes, one a decision node and one an event node, after alternative branches for testing in one or two markets. Because the cost of test marketing is $10,000 per market tested, we place $10,000 and $20,000 cost gates (written as 10 and 20) on the test in one market and test in two markets alternative branches respectively.

The beginning of the tree is shown in Figure 9.1, on which two continuations are indicated. The first of these occurs at the decision node for choosing between Lethbridge, Peterborough, and St. John's as the solitary test market. After each of these comes a similar structure, with only some of the numbers being different. First, there is a result event, with the test campaign in every city being either a success or a failure. There is a payoff of $30,000 associated with a success, and a payoff of $3000 associated with a failure. Since it is implied in the case that a failure in a test market would immediately end the venture, all we need do is put a '3' (for $3000) at the end of every outcome branch which represents failure. However, every 'success' outcome branch is treated differently. Because each of these is followed by more tree structure, we handle the $30,000 in revenue by using a payoff node followed by a null branch. On the null branch we place a cost gate, with the figure placed in parenthesis indicating that we have a revenue rather than a cost. Hence a revenue of \$30,000 is indicated as:



(30)

When the tree is rolled-back, the '30' is *added* to the number on the right of the null branch to obtain the number at the payoff node. After every null branch comes a decision about the national campaign, and then a result event if the 'proceed' alternative is followed. The tree structure of the first continuation is shown in Figure 9.2.

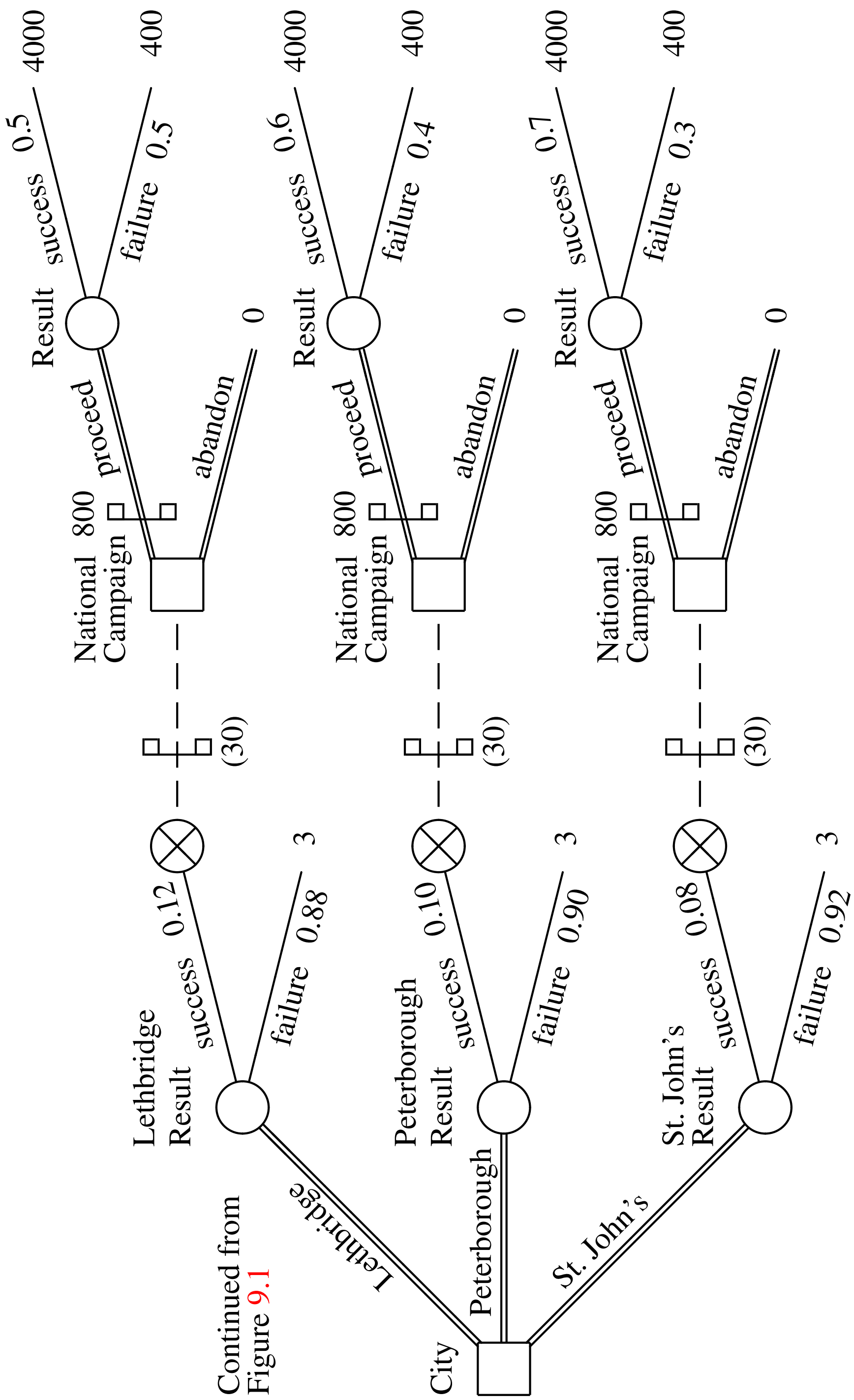


Figure 9.2: New Detergent Case: First Continuation

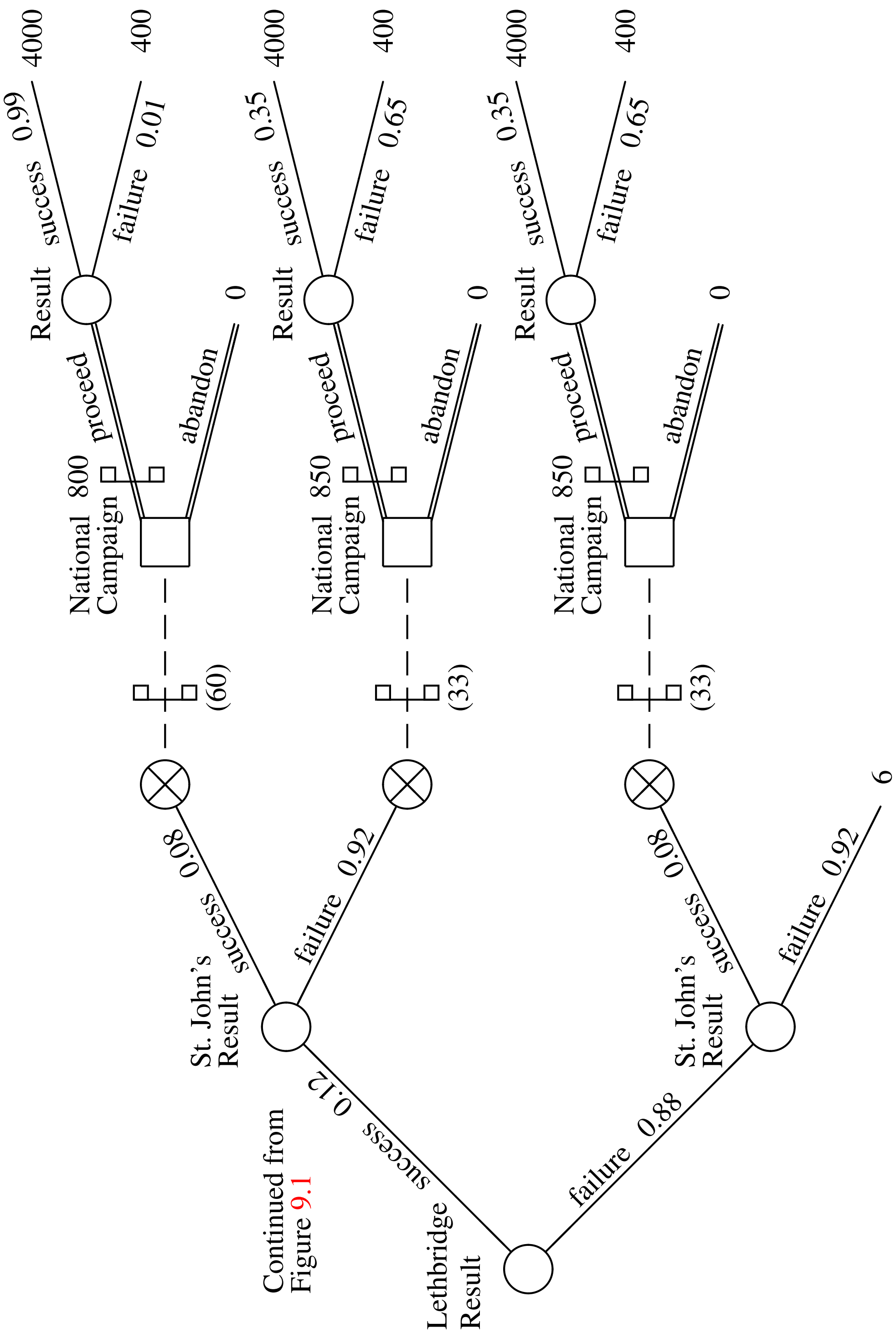


Figure 9.3: New Detergent Case: Second Continuation

The second continuation comes after the alternative of testing in two markets. Since the places of these markets have been stated in the case as being Lethbridge and St. John's, we next have the result events for these two markets. Here is an example where the order does not matter – it can either be the Lethbridge result event followed by the St. John's result event, or *vice versa*. In real life, these results would be announced more-or-less simultaneously. This is why the payoffs are combined – for example, if we are successful in both markets, then \$60,000 in revenue (i.e. \$30,000 from each place) is obtained. If one is a success, but the other is a failure, then \$30,000 + \$3000 = \$33,000 is obtained. Finally, if both are failures then the revenue is \$6000 (i.e. \$3000 from each place).

The rest of the tree is similar in structure to the first continuation, but we note that the cost of a national campaign after one failing test market is now \$800,000 + \$50,000 = \$850,000. The second continuation of the tree is shown in Figure 9.3.

9.1.4 Solution and Recommendation

We perform the rollback beginning with the first continuation. This is shown on Figure 9.5. Next, we perform the rollback for the second continuation. This is shown on Figure 9.6. At the extreme left, we obtain the figure 188.2224. The question arises as to how many decimal places we should report. Because the figures are in thousands of dollars, this figure represents \$188,222.40, so at least we aren't trying to report a fraction of a cent. Even so, some would argue that it's pretentious to report any figure closer than say the nearest ten dollars. My preference is to do things accurately, and then round the final answer, should that be desirable. It turns out in this example that the final answer is unaffected by this figure anyway.

The figures from the extreme left of the first and second continuations are then transferred to the initial part of the tree. That part of the tree is then rolled back, as shown in Figure 9.4.

Recommendation Make the ads for a test market campaign, and run this campaign in Peterborough. If this turns out to be a success, then proceed with the national campaign. If the test campaign turns out to be a failure, then abandon the project. The ranking payoff is \$156,700.

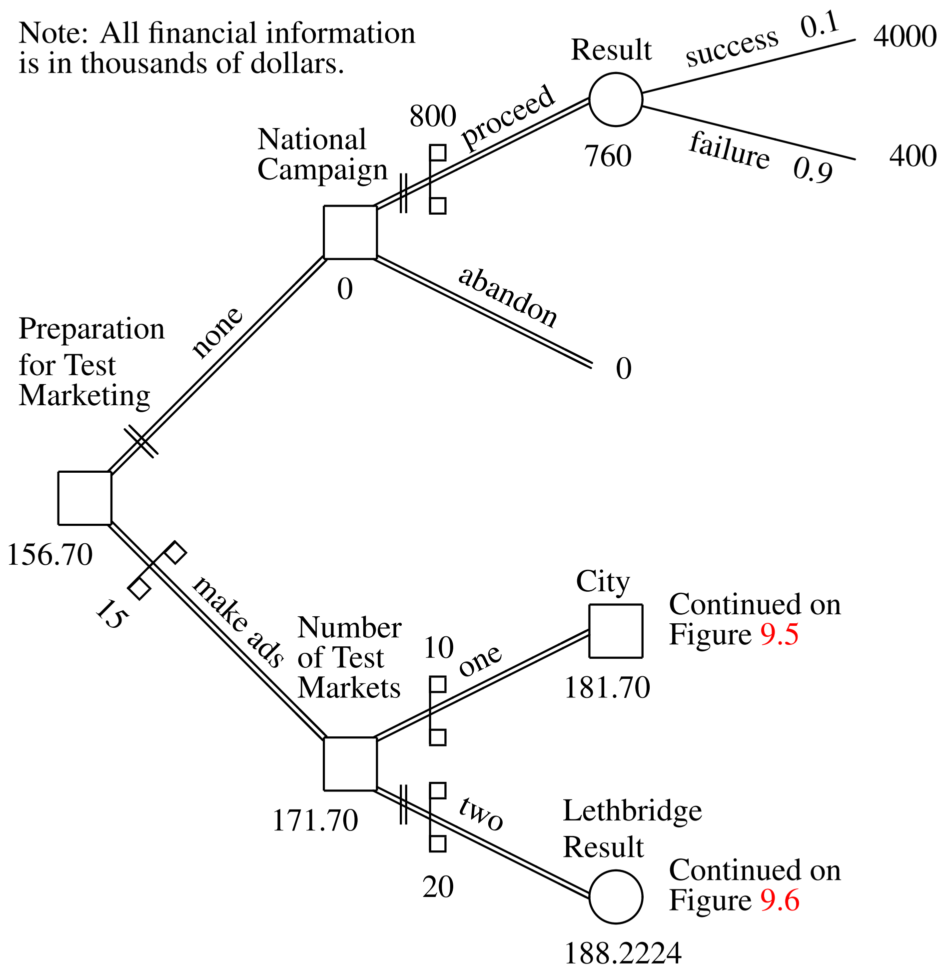


Figure 9.4: Rollback of the Beginning of the Tree

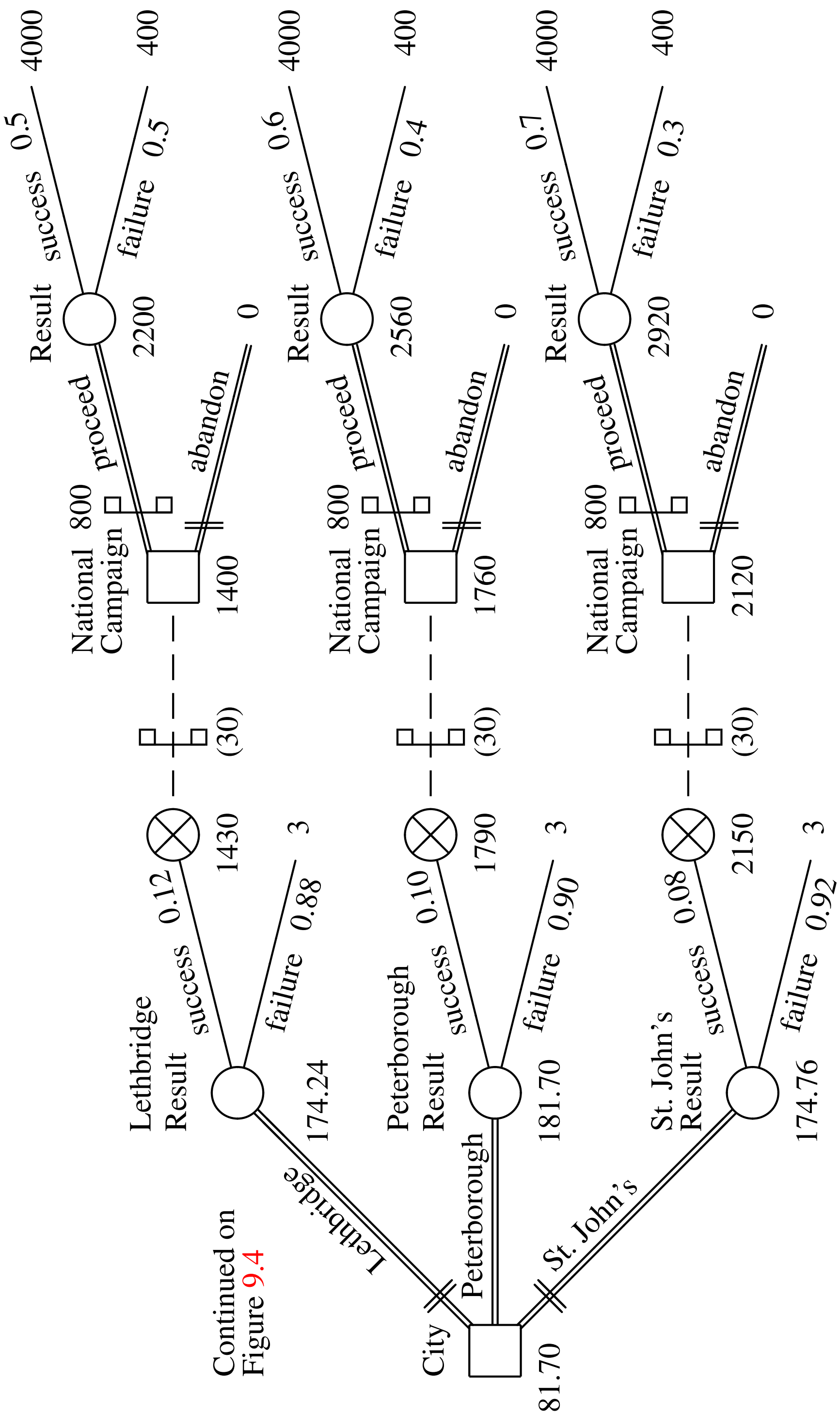


Figure 9.5: Rollback of the First Continuation

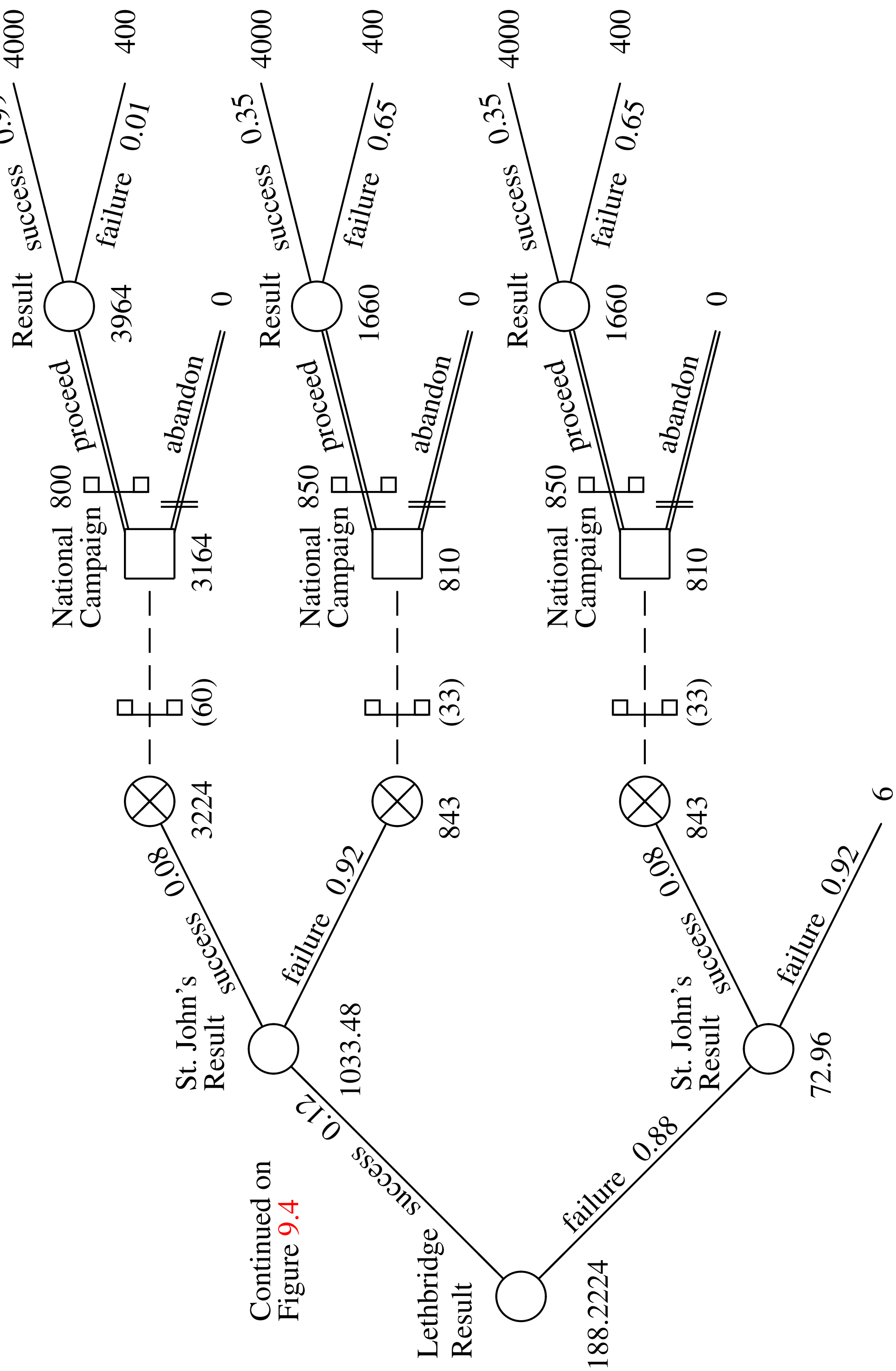


Figure 9.6: Rollback of the Second Continuation

9.2 Decision Trees without Revenues

In this section we consider an example which contains only costs.

9.2.1 Airline Ticket: Problem Description

An office manager in St. John's has been informed that a compulsory company-wide meeting might need to be held in Vancouver in fifteen days time. At the present time, there is about a 30% chance that the meeting will go ahead. There is about a 40% chance that in about five days from now they will know for sure whether or not the meeting will be held. If they still are not sure at that point, then there's still, as there is now, only a 30% chance that the meeting will go ahead. There is a 100% chance that in ten days time they will know for sure about the meeting one way or the other.

A full-fare economy return ticket, which would cost $3500, could be purchased as late as the day of the trip. Another option would be to buy a non-refundable ticket for $1300 which must be purchased at least seven days before departure. Another choice is to buy a non-refundable seat-sale ticket for $800, which would have to be purchased no later than tomorrow. Assuming that a non-refundable ticket would be worthless should the meeting not go ahead, develop and solve a decision tree to analyze the manager's problem.

9.2.2 Formulation

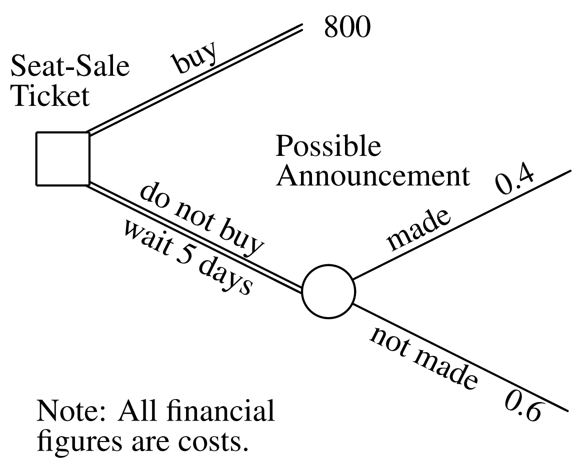
This example only mentions costs, not revenues, so if we put all the numbers onto the tree as we did in the previous section we will be rolling back negative numbers. Instead of dealing with negative numbers, we could write costs on the tree as positive numbers, and rollback the tree as before, except that at each square, we would choose the alternative with the *lowest* cost. We will solve this problem using this alternate approach; the final tree will have all financial information being the absolute value of what we would have had if we had not used this approach.

While there is a fifteen day continuum of time in this problem, only certain points in time are relevant. If we call today day 0, then there is the possibility of more information on day 5; if there's no announcement on day 5, then there will be an announcement on day 10. To attend a meeting on day 15, the manager must fly across the country no later than day 14. Then there are the deadlines for the purchase of the various classes of tickets: day 1 for the $800 ticket; day 7 for the

$1300 ticket; and day 14 for the full-fare $3500 ticket. Hence our focus should be on days 1, 5, 7, 10, and 14.

While the $3500 ticket can be purchased at any time, there is no advantage to purchasing it early. If the ticket is bought early, then a few days extra interest is charged, and more importantly, there would be the hassle of returning the ticket should the trip become unnecessary. If the few days interest is not important, then the ticket could be bought after day 10. There is no sense in buying a $1300 ticket today or tomorrow, because an $800 ticket is available during this time with the same privileges. After tomorrow, the office manager might as well wait until at least the end of day 5 to possibly obtain more information. Hence the $800 seat-sale ticket would be bought on days 0 or 1 (or not at all), the $1300 7 day advance ticket would be bought on days 6 or 7 (or not at all), and the $3500 full-fare ticket would be bought on days 11 to 14 inclusive (or not at all).

At the outset, the manager could buy an $800 ticket, or he could wait five days for more information. Hence the tree begins with two alternatives; there are no further branches after the alternative branch to buy a seat-sale ticket for $800. Because this is a final branch, and because we are writing costs as positive numbers, we do not need a cost gate – all we need to do is write 800 to the right of the branch. The wait five days option, however, then has an event with two outcomes: an announcement is made; or no announcement is made. The tree so far is:



After the outcome branch for the announcement being made there is an event with two outcomes: the meeting will go ahead, or it will not go ahead. It is possible, but not advisable, to combine the two events into one event with three outcomes: the meeting will go ahead; it will not go ahead; and no announcement. Doing it this way would shorten the tree, but it would require computing some joint probabilities – this confuses the formulation process with the solution process. We will therefore write what is happening as two events.

After the "will go ahead" outcome branch, we could have a square with two alternatives, one for buying a 7 day advance ticket, and one for buying a full-fare ticket for \$3500. However, it is obvious that the manager should buy a 7 day advance ticket for \$1300, so we will only draw this alternative. After the "will not go ahead" outcome branch, no action needs to be taken, so we simply write a payoff of 0 to the right of this branch.

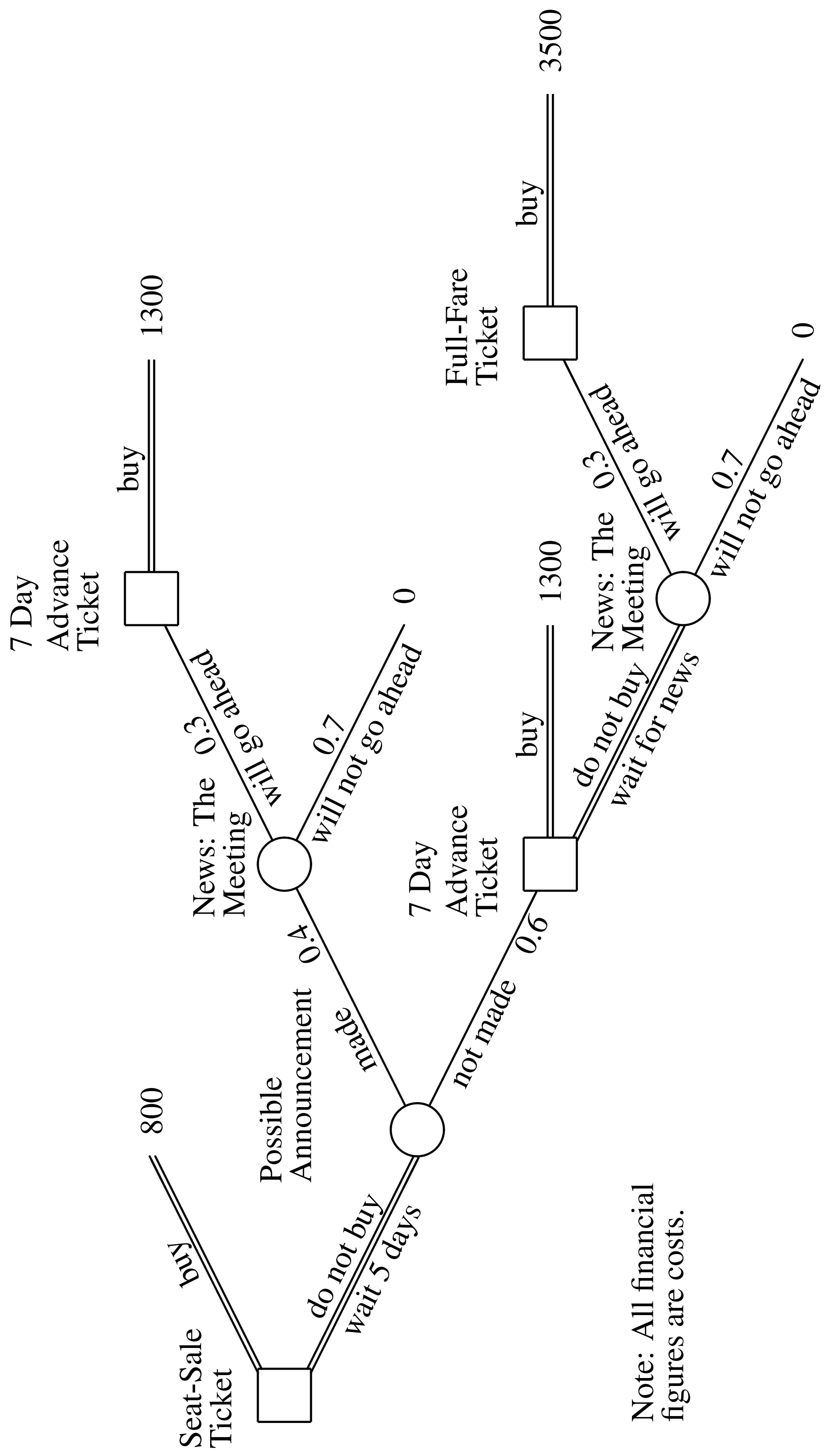


Figure 9.7: Airfare Problem before Rollback

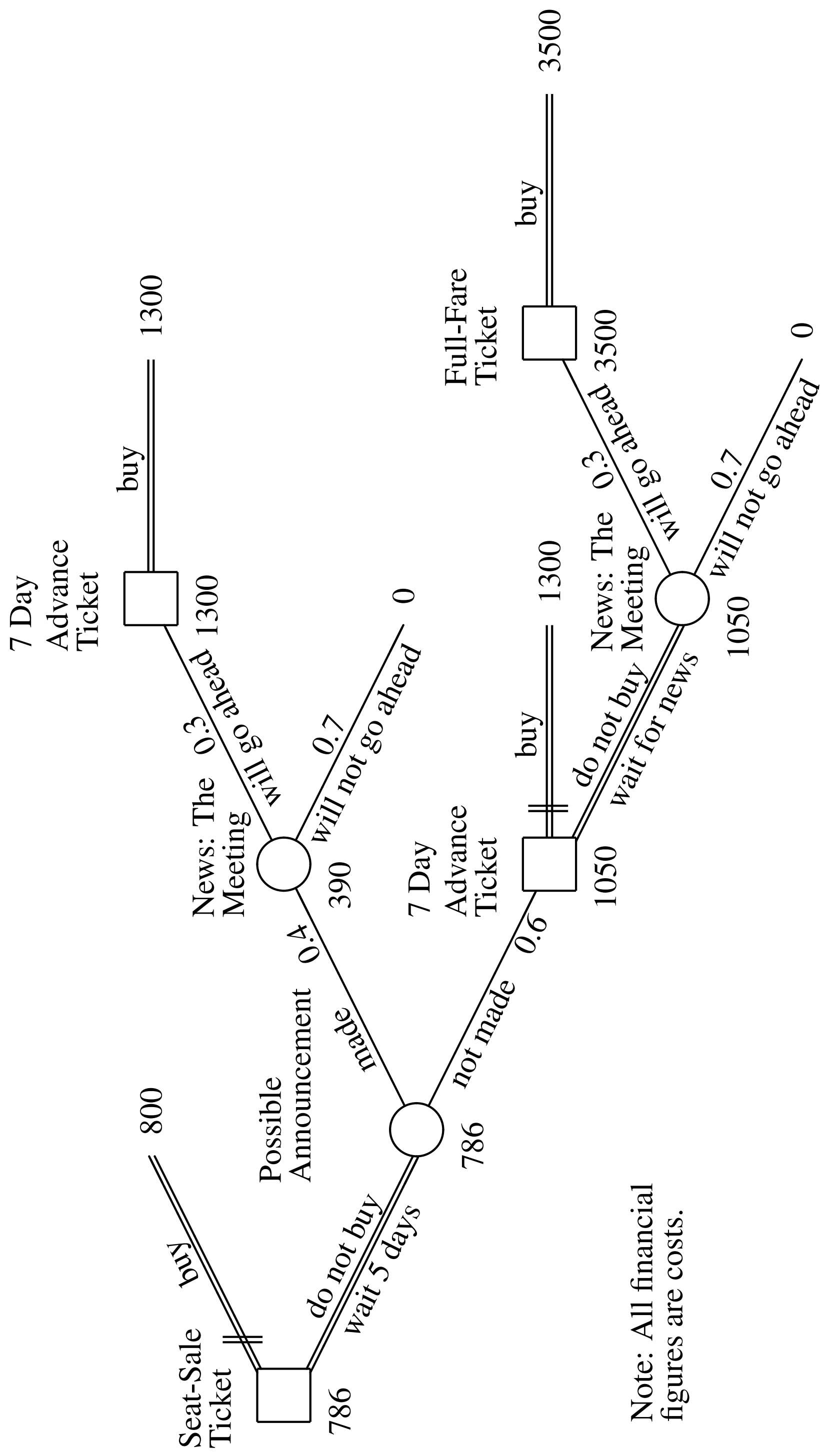


Figure 9.8: Airfare Problem after Rollback

If there's no announcement after five days, then our choices are to either buy a 7 day advance ticket or wait another five days. If the latter is chosen, then an event occurs giving information about the meeting. After the "go ahead" branch, the manager must buy a full-fare ticket; after the "will not go ahead" branch, no action is required.

The entire tree, printed in landscape form, appears in Figure 9.7. Because the financial information is all costs, a note has been placed on the figure to that effect.

9.2.3 Solution

To rollback the tree, we need to choose the lowest cost at each square. With this modification, the rolled-back tree appears in Figure 9.8. The recommended course of action can be followed on the tree, but the analyst should also make the recommendation by clearing stating it in words. In trees with multiple decisions, we often use the term *ranking profit* (or *ranking cost* in this example) to indicate that the number being presented is a mixture of measures (best at a square, expected value at a circle). This is reported along with the best course of action.

9.2.4 Recommendation

Do not buy the $800 seat-sale ticket, but instead wait to see if there's an announcement in five days time. If there's an announcement that the meeting will go ahead, then buy a $1300 7 day advance ticket at that time. If there's an announcement that the meeting is not going ahead, then do nothing. If there's no announcement after five days, then wait for a further announcement. If the meeting is going ahead, then buy a $3500 full-fare ticket; otherwise, do nothing. The ranking cost is $786.

Although the $786 figure is the most important one on the tree, the other rolled-back numbers are also important, because they give the ranking cost to be incurred for proceeding further down that path. For example, if an announcement is not made after five days, then the ranking cost increases from $786 to $1050.

9.2.5 The EVPI

In this example we need to find the expected *cost* with perfect information (EC with PI). If at the outset we were to receive perfect information that the meeting will be going ahead, then we would buy the seat-sale ticket for $800, otherwise we would do nothing. The chance that we will be told that the meeting will be going ahead is 30%, hence EC with PI = 0.3(800) + 0.7(0) = \$240. The expected cost without information is \$786. We subtract to find the EVPI, in reverse order because these are costs.[1]

$$\begin{aligned} \text{EVPI} &= \text{EC without PI} - \text{EC with PI} \\ &= \$786 - \$240 \\ &= \$546 \end{aligned}$$

9.3 Decision Making with Bayesian Revision

9.3.1 Introduction

Bayesian revision is a procedure for determining conditional probabilities in the reverse order to which they are initially known. In this and the next section, we examine decision trees for which the use of Bayesian revision is needed in order to compute some of the probabilities. Starting with a problem description, we begin to develop the decision tree, except that not all of the probabilities can be written down immediately. We then perform a Bayesian revision to find these probabilities, and then transfer these numbers to the decision tree. The tree is then rolled-back to obtain a recommendation for the situation.

There are two methods for performing Bayesian revision. One way involves making three tables. The table method is what should be used if these calculations are to be performed on a spreadsheet. A second method involves making what are called *prior* and *posterior* trees. This is a visual approach suitable to computations by hand. We will show both these methods in detail.

9.3.2 Seismic Testing Problem Description

An oil exploration company has identified a site under which there may be a pocket of oil. The probability that oil exists at this location is 1%. It would cost \$3,000,000 to drill for oil. If the oil exists, it would be worth \$40,000,000. A seismic test is available which would cost \$40,000. The result of the test would be one of the following: “positive”, “inconclusive”, or “negative”. If there really is oil present, then there is a 60% chance of a positive reading, a 30% chance of an inconclusive reading, and a 10% chance of a negative reading. If there’s no

[1] This is the same as saying that the EVPI is $-\$240 - (-\$786) = \$546$.

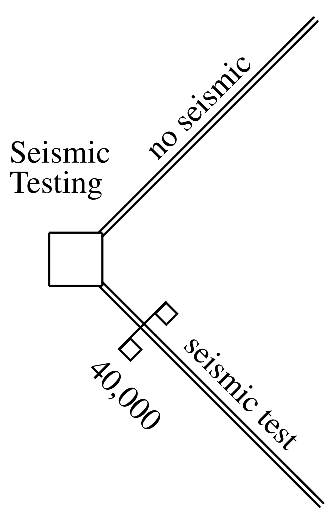
oil at that location, then there's a 0.04 probability of a positive reading, and a 0.2 probability of an inconclusive reading.

We wish to develop a decision tree for this situation, and solve it to obtain a recommendation for the oil exploration company.

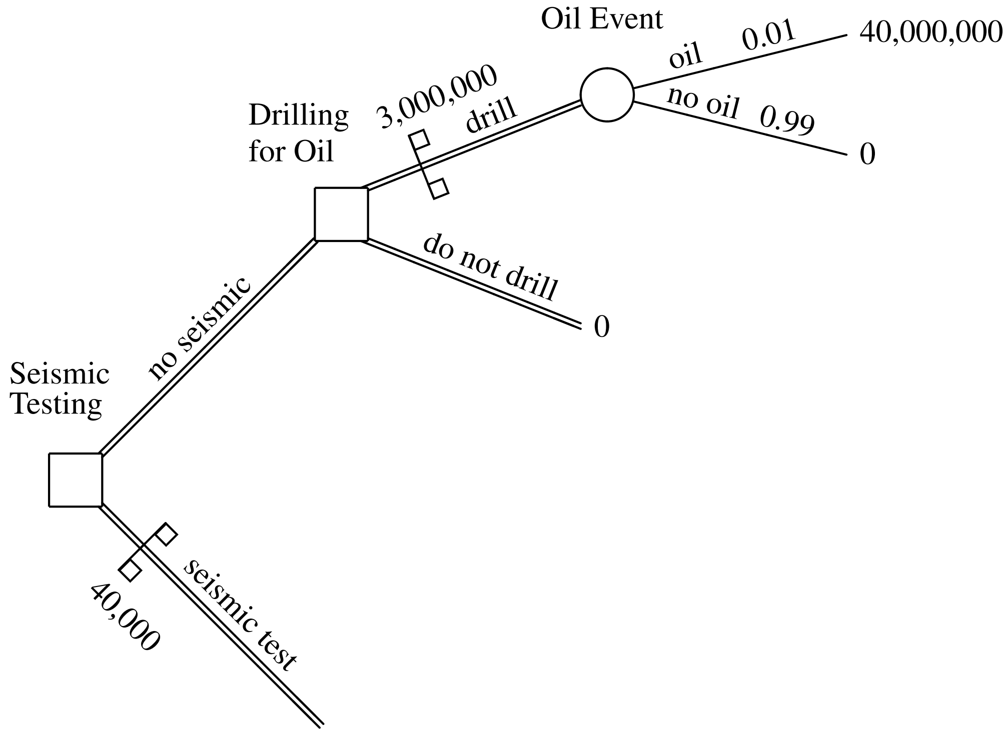
9.3.3 Problem Formulation

There are two decisions to be made in this situation. What we might call the major decision is whether or not to spend $3,000,000 drilling for oil. The other decision is whether or not to spend $40,000 to do the seismic test. The purpose of the seismic test is to obtain information which would help us with the major decision. We will call the decision about the seismic test the information decision.

Not just in this situation, but in all problems of this type, the information decision must precede the major decision. Indeed, the information decision precedes everything else. This decision, with its two alternatives, is as follows:



If the seismic test is not done, then this becomes an easy problem. We must choose whether or not to drill at a cost of $3,000,000, and if we drill we then have an oil event with two outcomes: oil is present with probability 0.01; and oil is not present with probability 0.99. Adding these things to the tree we obtain:

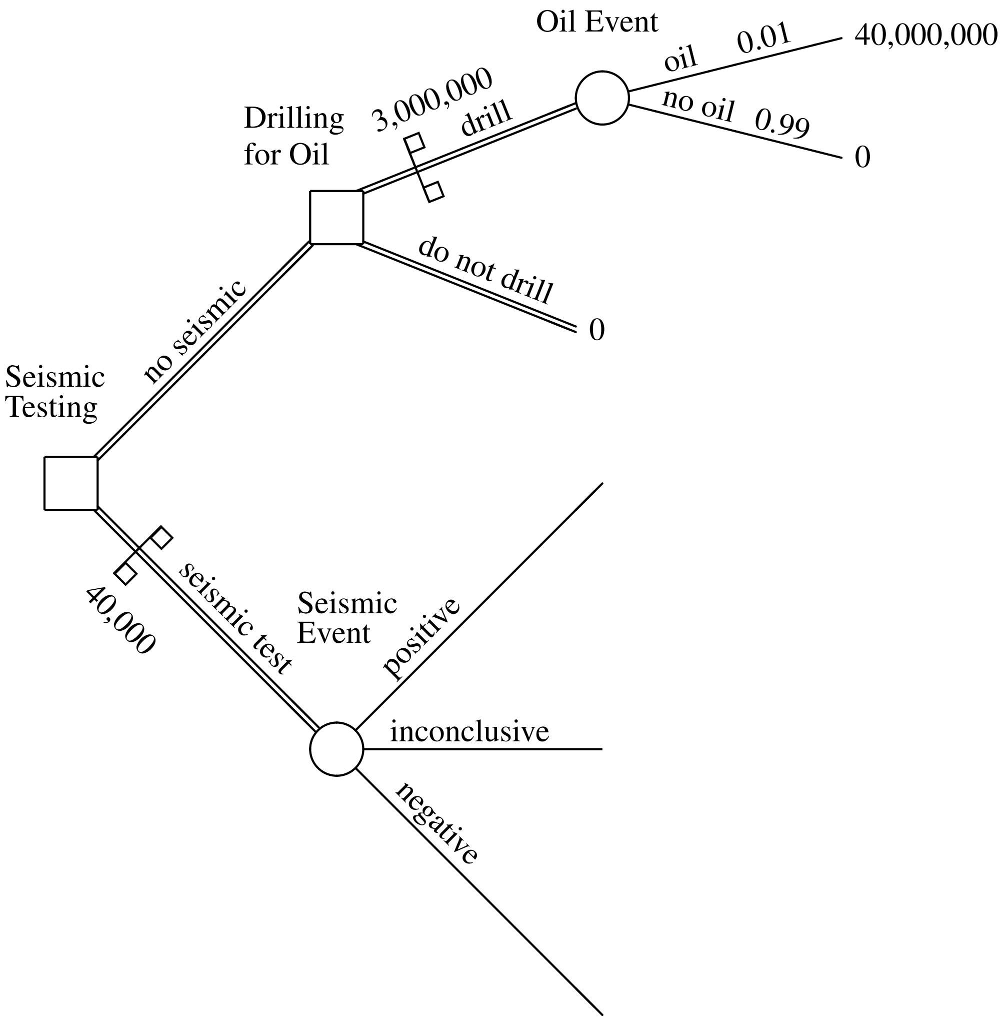


In problems of this type, it is sometimes useful to find the EVPI before considering the possibility of obtaining more information. Then, if it turns out that the cost of obtaining information is higher than the EVPI, then we can eliminate the alternative to seek information. We note that in the preceding tree, if we rollback the top part we would obtain \$400,000 at the Oil Event circle, and then \$0 at the Drilling for Oil square. Now suppose that we reverse the order of the decision and the event in order to find the EV with PI. In this case, the EV with PI is:

$$0.01(40,000,000 - 3,000,000) + 0.99(0) = 370,000$$

Since the EV without PI was \$0, the EVPI is also \$370,000. The cost of the seismic test, which is \$40,000, is much less than \$370,000. Hence, the seismic test cannot be trivially eliminated.

After the alternative to do the seismic test, comes the seismic test event, with its three outcomes: positive; inconclusive; and negative. This is an example of a common pattern in this type of problem – an alternative of the information decision for which information is sought is followed by an information event, which in turn is followed by the major decision. When we draw the outcome branches for this situation, we cannot immediately write the probabilities, for we do not know what they are. We will find them later using Bayesian revision, and will then transfer these numbers to the decision tree. Adding these outcome branches we obtain:



At this point, a fair bit of repetition appears in the rest of the tree. After each of the outcomes of the seismic test event, there is the decision about drilling.[2] If the drilling is done, it is followed by the oil event. This of course is like what

[2]Note that we consider the "drill" alternative even after a "negative" seismic test. This is because the information is not perfect, so there is a chance that doing the non-obvious thing may be right.

we have already drawn at the top of the tree, but there's one important exception. The probabilities of oil and no oil are not 0.01 and 0.99 as they were before. Instead, these are now conditional probabilities, and they must be calculated using Bayesian revision. The decision tree with the probabilities absent on the bottom part of the tree is shown in Figure 9.9. To reduce the clutter on this part of the tree, the words "Drilling for Oil" and "Oil Event" only appear once rather than in all three places.

9.3.4 Bayesian Revision

Now, we must do the Bayesian revision. We begin by showing the table method.

The event for which the marginal probabilities are known is that of the presence of oil. These probabilities are 0.01 for the existence of oil at that location, and 0.99 for the absence of oil. For the other event, the seismic testing, we have probabilities (given in the problem description) which are conditional on whether there is or is not oil in the ground. These probabilities, and the two marginal probabilities, are given in the following table. Note that since one of the three outcomes of the seismic test must occur, we find P(negative/no oil) as $1-(0.04+0.20)=0.76$.

		Seismic Event			
		positive	inconclusive	negative	
Oil	oil	0.60	0.30	0.10	0.01
Event	no oil	0.04	0.20	0.76	0.99

Multiplying the conditional probabilities by the marginal probabilities of the oil event we obtain the joint probabilities. Some of the joint probabilities require four places after the decimal, so all are shown this way so that they line up properly. Summing the joint probabilities in each column gives the marginal probabilities of the seismic test event. The second table is:

		Seismic Event			
		positive	inconclusive	negative	
Oil	oil	0.0060	0.0030	0.0010	0.01
Event	no oil	0.0396	0.1980	0.7524	0.99
P(seismic result)		0.0456	0.2010	0.7534	

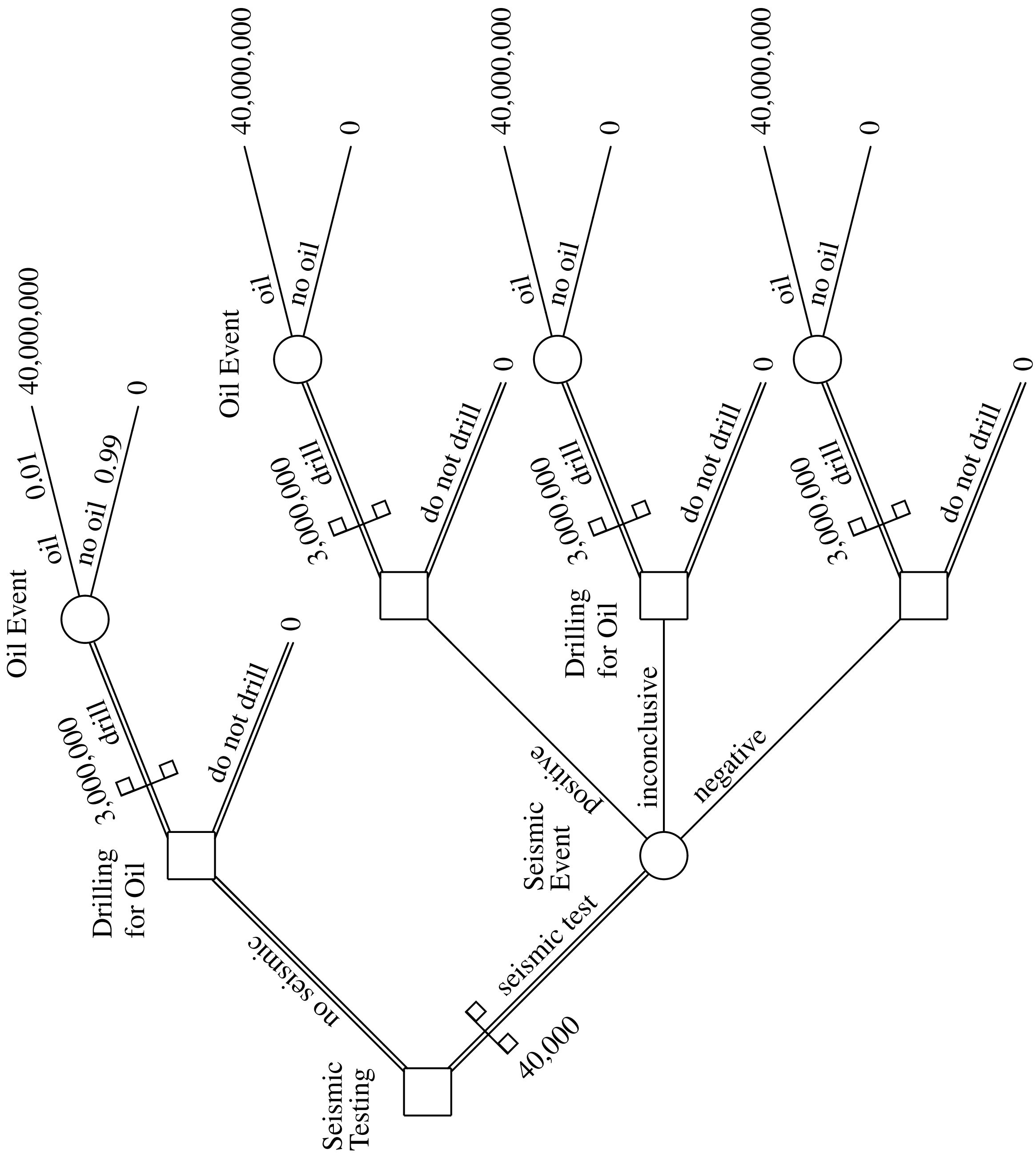


Figure 9.9: Seismic Testing – Decision Tree without Revised Probabilities

Finally, dividing the joint probabilities by the marginal probabilities underneath we obtain the posterior conditional probabilities. The third table using five-place decimals is:

		Seismic Event		
		positive	inconclusive	negative
Oil	oil	0.13158	0.01493	0.00133
Event	no oil	0.86842	0.98507	0.99867
P(seismic)		0.0456	0.2010	0.7534

We read this as P(oil/positive) = 0.13158, P(no oil/positive) = 0.86842, and so on.

These figures, even though there are five-place decimals, are approximations of exact fractions. If we wish, we can use fractions instead. If this is done, it makes sense to remove the decimals from the numerator and the denominator. However, it does not make sense to reduce the fraction to the lowest common denominator, as this only adds work. For example, instead of calculating the decimal quantity 0.01493, we could have expressed 0.0030 divided by 0.2010 as the fraction $\frac{3}{201}$, but we need not reduce this fraction to $\frac{1}{67}$. As unreduced fractions the third table is:

		Seismic Event		
		positive	inconclusive	negative
Oil	oil	$\frac{60}{456}$	$\frac{3}{201}$	$\frac{10}{7534}$
Event	no oil	$\frac{396}{456}$	$\frac{198}{201}$	$\frac{7524}{7534}$
P(seismic)		0.0456	0.2010	0.7534

The concern about accuracy may seem to be misplaced when all the original probabilities in the first table are approximations anyway. However, some of the probabilities will be multiplied by large numbers, specifically the \$40,000,000 figure. Here's what happens depending on the level of accuracy when we approximate $\frac{60}{456}$ using decimals. The decimal expansion is 0.13157947... When multiplied by \$40,000,000, we obtain (to the nearest cent) \$5,263,157.90. If we approximate the decimal we obtain (using rounding) 0.132 for three places, 0.1316 for four places, and 0.13158 for five places. The values of these numbers times \$40,000,000, and the differences between these values and the theoretical value are:

Value	Value × \$40,000,000	Variation
0.13157947...	\$5,263,157.90	–
0.13158	\$5,263,200.00	\$42.10
0.1316	\$5,264,000.00	\$842.10
0.132	\$5,280,000.00	\$16,842.10

These variations are what would be present at the "Oil Event" node which comes after a "positive" outcome for the seismic test. By the time everything is rolled back, the error would be diminished, but it would still be considerable. Such errors can be avoided by storing all probabilities in the calculator's memory, so that the nearly exact value is used, even if only five decimal places are written out in full – doing it this way is equivalent to using fractions. For student use, either doing it that way or the use of five decimal places (rounded) is recommended. At the very least, one should use four decimal places (rounded); using only three can cause substantial errors.

A big advantage of the table method over the method of prior and posterior trees, is that the table method is easily adapted to Excel. Here is the table method, showing the formulas (entered into C9, C11, C15, and then copied) needed to compute the numbers in the second and third tables:

	A	B	C	D	E	F
1						
2				**Seismic Event**		
3			Positive	Inconclusive	Negative	**Prob.**
4	**Oil**	Oil	0.6	0.3	0.1	0.01
5	**Event**	No oil	0.04	0.2	0.76	0.99
6						
7				**Seismic Event**		
8			Positive	Inconclusive	Negative	**Prob.**
9	**Oil**	Oil	=C4*$F4	=D4*$F4	=E4*$F4	0.01
10	**Event**	No oil	=C5*$F5	=D5*$F5	=E5*$F5	0.99
11		**Prob.**	=SUM(C9:C10)	=SUM(D9:D10)	=SUM(E9:E10)	
12						
13				**Seismic Event**		
14			Positive	Inconclusive	Negative	**Prob.**
15	**Oil**	Oil	=C9/C$11	=D9/D$11	=E9/E$11	0.01
16	**Event**	No oil	=C10/C$11	=D10/D$11	=E10/E$11	0.99
17		**Prob.**	=C11	=D11	=E11	

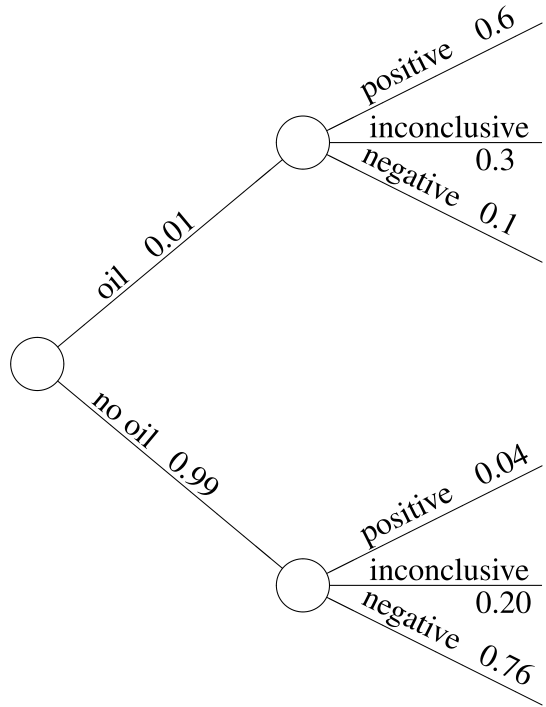
The numerical values are:

	A	B	C	D	E	F
1						
2				**Seismic Event**		
3			Positive	Inconclusive	Negative	**Prob.**
4	**Oil**	Oil	0.60	0.30	0.10	0.01
5	**Event**	No oil	0.04	0.20	0.76	0.99
6						
7				**Seismic Event**		
8			Positive	Inconclusive	Negative	**Prob.**
9	**Oil**	Oil	0.0060	0.0030	0.0010	0.01
10	**Event**	No oil	0.0396	0.1980	0.7524	0.99
11		**Prob.**	0.0456	0.2010	0.7534	
12						
13				**Seismic Event**		
14			Positive	Inconclusive	Negative	**Prob.**
15	**Oil**	Oil	0.131579	0.014925	0.001327	0.01
16	**Event**	No oil	0.868421	0.985075	0.998673	0.99
17		**Prob.**	0.0456	0.2010	0.7534	

Before transferring these probabilities to the decision tree, we will look at the prior and posterior tree method of performing Bayesian revision. This method takes a little bit longer to do, but it's conceptually easy because it mimics a subset of the decision tree. We work with two probability trees, called the *prior* tree and the *posterior* tree.

Both the prior and posterior trees contain two events. The prior tree (which is done first) has the two events of the decision tree in the reverse order of how they appear in the decision tree. The posterior tree (which is done after completing the prior tree) has the two events in the reverse order of how they appear in the prior tree. Equivalently, the posterior tree has the two events in the same order as they appear in the decision tree.

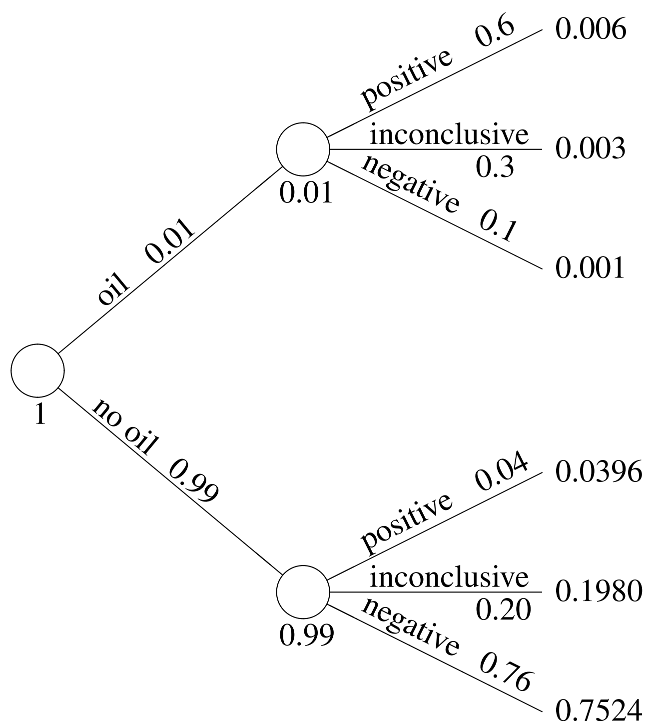
For this example, the two events of the decision tree are the seismic test event and the oil event, in that order. Since the prior tree contains these events in reverse order, the prior tree consists of the oil event followed by the seismic test event. Writing the outcomes of the oil event with their marginal probabilities, and the three outcomes of the seismic test event with their conditional probabilities, gives us the following picture.



On this tree we write the joint probabilities at each node and at the ends of the branches. The node on the left begins with a probability of 1, meaning that it is certain that something will occur. For any outcome branch on the tree, the joint probability at the ending node (on the right) is the joint probability at the beginning node (on the left) multiplied by the probability (be it marginal or conditional) on that outcome branch. For example, the joint probability at the top seismic test event node is 1 (the joint probability at the oil event node) multiplied by 0.01 (the marginal probability along the "oil" outcome branch, which is simply 0.01). Similarly, the joint probability at the bottom seismic test event node is 0.99. So far, everything is trivial.

The joint probability at the end of the top "positive" branch equals 0.01 (the joint probability at the seismic test event node) multiplied by 0.6 (the conditional probability along the "positive" outcome branch), which is 0.006. Similarly, the joint probabilities at the end of the top "inconclusive" and "negative" outcome branches are $0.01(0.3) = 0.003$ and $0.01(0.01) = 0.001$ respectively. The joint probability at the end of the bottom "positive" branch equals 0.99 (the joint probability at the seismic test event node) multiplied by 0.04 (the conditional probability along the "positive" outcome branch), which is 0.0396. Similarly, the joint probabilities at the end of the bottom "inconclusive" and "negative" outcome branches are $0.99(0.20) = 0.1980$ and $0.99(0.76) = 0.7524$ respectively.

Comparing this approach with the table method, it is seen that the prior tree is simply a visual way of displaying the information which appears in the first table and in part of the second table. Adding the joint probabilities the completed prior tree is:

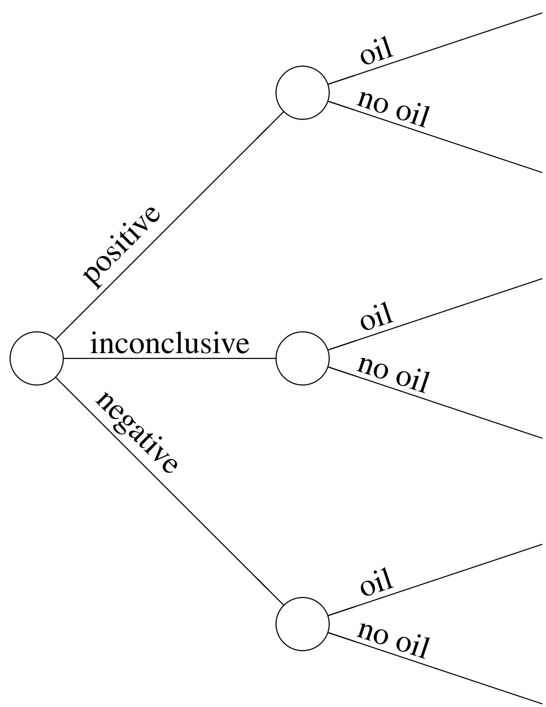


The sum of the joint probabilities on the extreme right of the tree must sum to 1, so it is wise to verify this fact before proceeding to the posterior tree.

$$.006 + .003 + .001 + .0396 + .1980 + .7524 = 1.000 \quad \surd$$

This sum doesn't have to be written out as it is here, but a calculator should be used to verify that sum is 1. Whenever the sum is not 1, it means that an error has been made, which needs to be corrected before proceeding further.

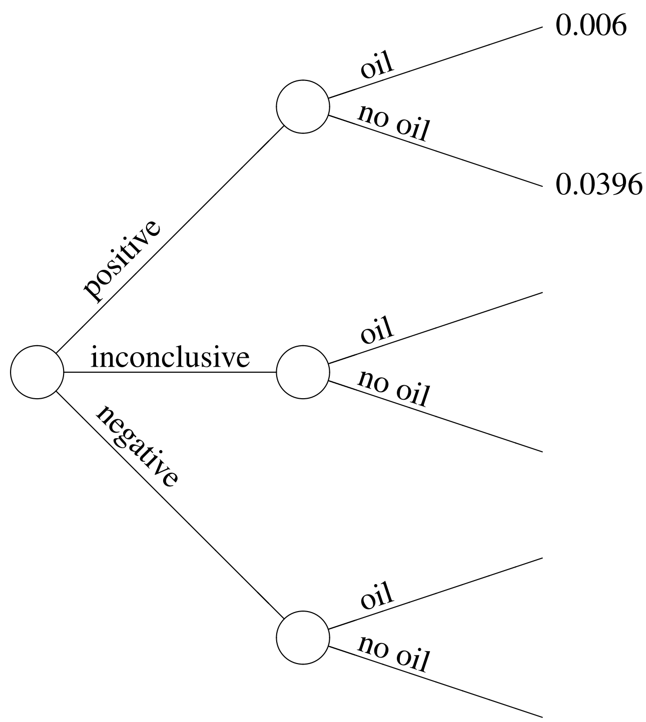
As stated earlier, the posterior tree contains the same events, but in reverse order. For this problem, the posterior tree begins with the seismic test event, which is followed by the oil event. We begin drawing the posterior tree by outlining its shape; the probabilities need to be computed by transferring the final joint probabilities from the prior tree. The shape of the tree is:



Throughout the development of the decision, prior, and posterior trees, we have maintained consistency in the vertical ordering of the outcomes. We have always placed “oil” above “no oil”, and “positive” above “inconclusive” which in turn is above “negative”. This consistency will help us when transferring the joint probabilities from the prior tree to the posterior tree, and when transferring marginal and conditional probabilities from the posterior tree to the decision tree.

The final joint probabilities on the prior and posterior trees are the same, except that they are placed in a different order. The first (top) joint probability on the posterior tree is the joint probability of "positive" and "oil". This is numerically the same as the joint probability of "oil" and "positive", which is found on the prior tree (at the top), and its value is 0.006.

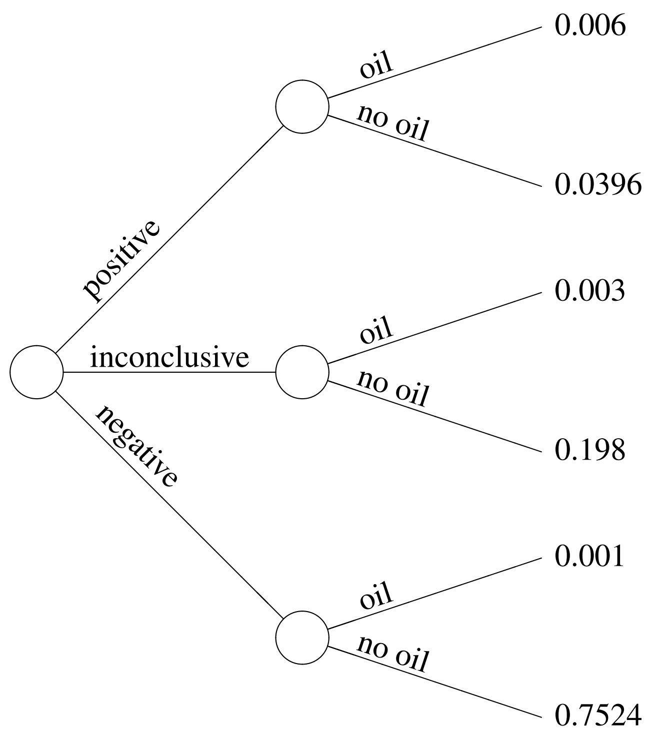
The second (from the top) joint probability on the posterior tree is the joint probability of "positive" and "no oil". This is numerically the same as the joint probability of "no oil" and "positive", which is found on the prior tree (fourth from the top), and its value is 0.0396. Placing these values on the posterior tree we have:



Going to the third place, we need the joint probability of "inconclusive" and "oil", which from the prior tree is seen to be 0.003. Because of the consistent vertical labelling of the outcomes, a nice pattern emerges. The top three joint probabilities on the prior tree become the joint probabilities at the top of each pair of outcomes on the posterior tree, and the bottom three joint probabilities on the prior tree become the joint probabilities at the bottom of each pair of outcomes on the posterior tree.

Though the patterns differ from one Bayesian revision to the next, there will always be a type of pattern to find whenever the outcomes have been labelled consistently. If there is any doubt, remember that for each joint probability the words on the outcomes of the prior and posterior trees must match up (in reverse order).

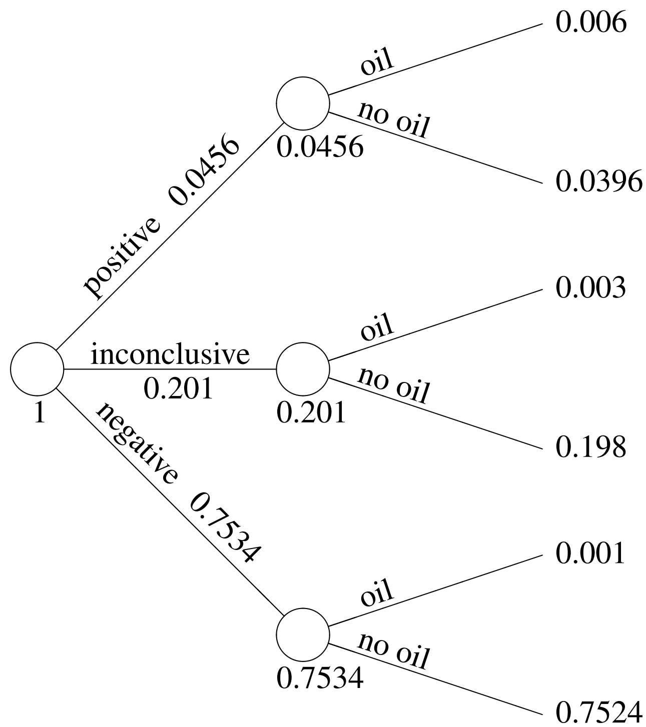
Completing the remaining four joint probabilities the posterior tree becomes:



After transferring the ending joint probabilities from the prior tree, the next step is to compute the other joint probabilities on the posterior tree. This is done simply by addition. At the top oil event node, we add 0.006 and 0.0396 to obtain 0.0456, at the middle node we add 0.003 and 0.198 to obtain 0.201, and at the bottom the sum of 0.001 and 0.7524 is 0.7534. Each of these numbers is written next to its corresponding event node. Then, taking the numbers we have just computed, we sum them to obtain

$$0.0456 + 0.201 + 0.7534 = 1.0000$$

This number 1 is written next to the left-hand node. It is always true that we should obtain a 1 next to this node, so this acts as a check on our calculations. Had we not obtained a 1, this would have indicated that an error had been made. The probability on every outcome branch is obtained by dividing the ending (right-side) joint probability by the beginning (left-side) joint probability. For the outcomes on the left-side event, these are just the ending joint probabilities divided by 1, producing the same numbers. Doing this much the posterior tree becomes:

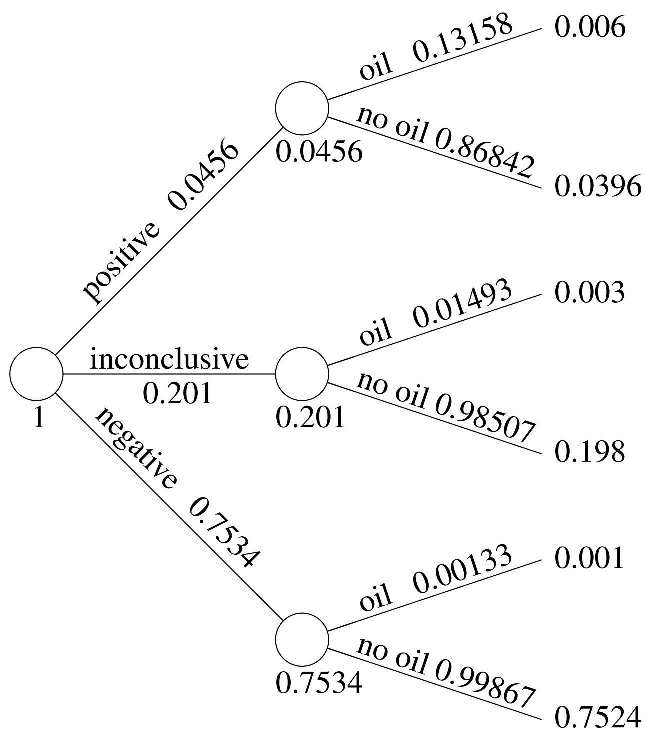


At this point we have found all the information in the second table of the table method of Bayesian revision. We now complete the Bayesian revision on the posterior tree, which provides the information found on the third table.

We continue the process of dividing joint probabilities, which provides the conditional probabilities. Starting with the top oil event (which comes after a "positive" outcome for the seismic test), the conditional probability of oil is computed as

$$\text{P(oil/positive)} = \frac{0.006}{0.0456} \approx 0.13158$$

As stated earlier, we may wish to give the exact value by writing the conditional probability as an unreduced fraction, i.e. $\frac{60}{456}$. Writing all six conditional probabilities rounded to five decimal places the posterior tree becomes:



While we have developed the prior and posterior trees slowly to illustrate the process, when doing this in practice all that is needed is a single sheet of paper on which both trees are written. This is shown in Figure 9.10.

This methodology for performing Bayesian revision is entirely optional. However, the reader should try both methods on a couple of problems before making this decision. When one becomes used to both methods, there really isn't much difference in time.

The main advantage of the table approach is that it can be done on a spreadsheet (though this is of no help on a test). The main advantage of the tree method is that it ties in nicely with the decision tree for which the Bayesian revision is being performed. Once the posterior tree has been completed, it is very easy to see where to transfer the marginal and conditional probabilities onto the decision tree.

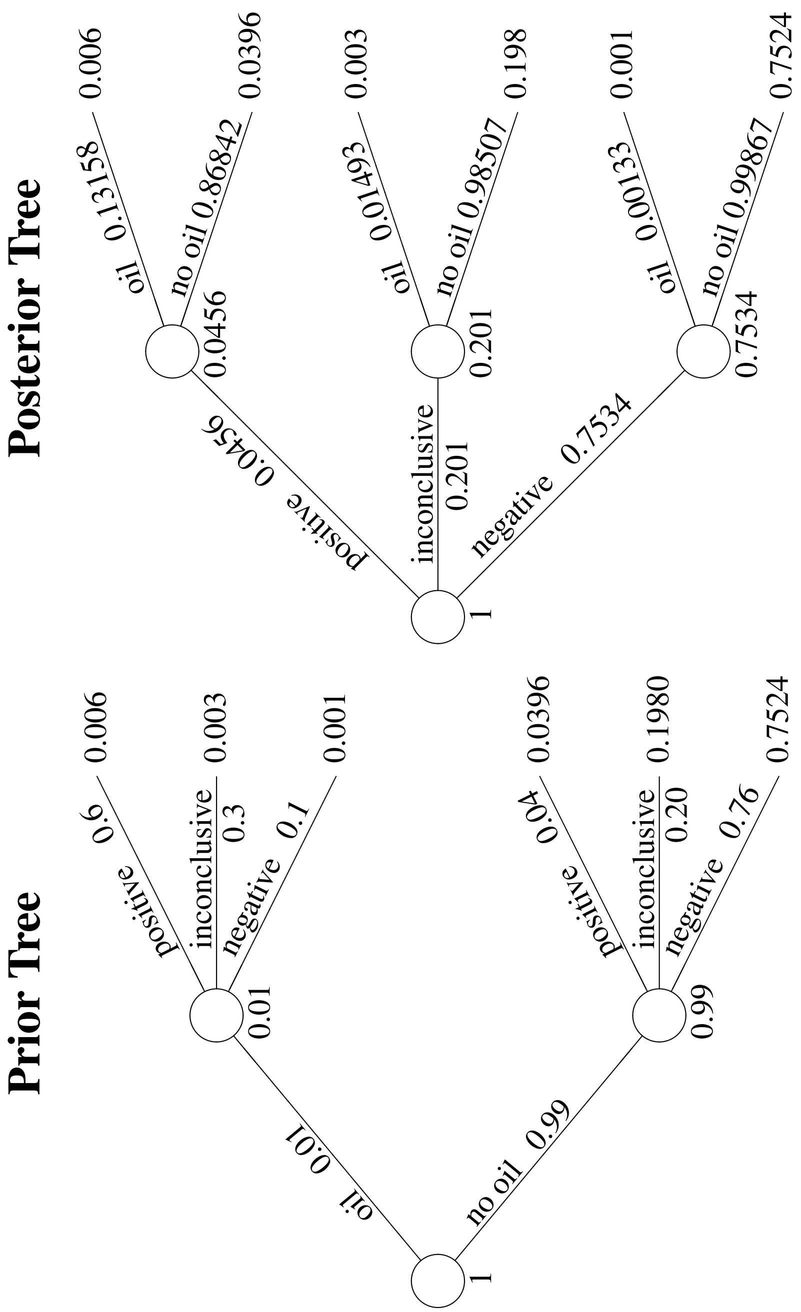


Figure 9.10: Prior and Posterior Trees for the Oil Drilling Problem

To review, the steps involved in making the prior and posterior trees are:

1. Taking the events from the decision tree in reverse order, make the prior tree showing its shape, putting labels on the outcomes, and write the marginal and conditional probabilities.

2. Using multiplication, find all the joint probabilities on the prior tree, and verify that they sum to 1.

3. Taking the events from the prior tree in reverse order, make the posterior tree showing its shape, and put labels on the outcomes.

4. Transfer the final (right side) joint probabilities from the prior tree to the appropriate places (i.e. matching pairs of outcomes) on the right side of the posterior tree.

5. Using addition, find the other joint probabilities, and verify that the initial (extreme left side) joint probability is 1.

6. For every outcome branch, find the probability on the branch by dividing the joint probability at the end of the branch by the joint probability at the beginning of the branch.

9.3.5 Solution and Recommendation

Now we can complete the formulation of the decision tree. Again, the consistency in the vertical labelling of the outcomes makes the transfer of the marginal and conditional probabilities from the posterior tree to the decision tree very easy. With Figures 9.9 and 9.10 in hand, we can easily see what needs to be transferred where. The events are in the same order, the difference being that in the decision tree there is a decision between the two events.

There are a total of three marginal probabilities, and six conditional probabilities, to be transferred from the posterior tree to the decision tree. (Alternatively, if the table method is used, we transfer these numbers from the third table.) Doing this, we obtain the decision tree shown in Figure 9.11.

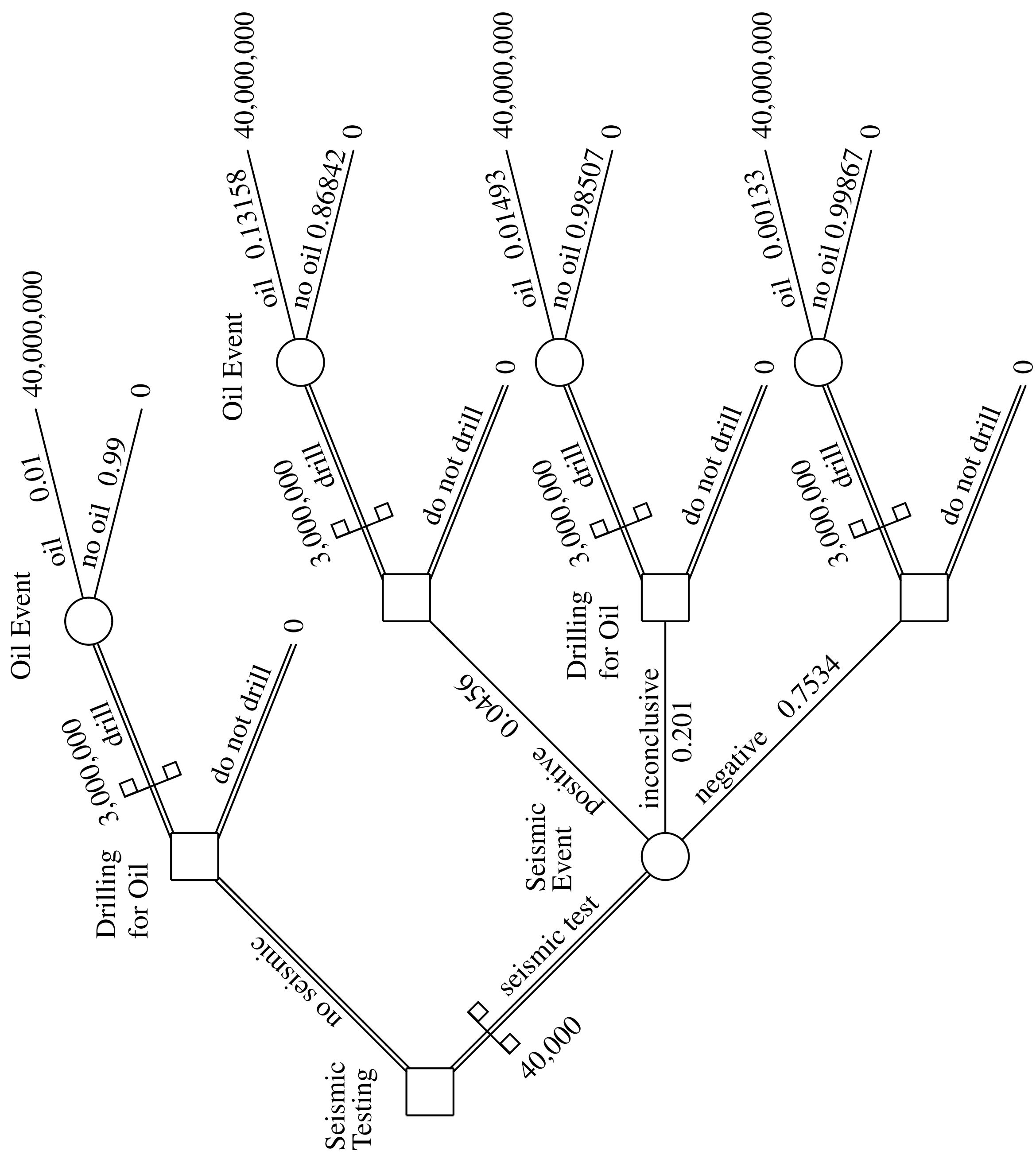


Figure 9.11: Seismic Testing – Decision Tree with Revised Probabilities

This tree is then rolled-back to obtain a recommendation. The rolled-back tree is shown in Figure 9.12. The conditional probabilities are shown to five-place accuracy, and the rolled-back payoffs are shown to the nearest dollar, but in fact all this information was stored to the accuracy of the calculator.

Recommendation Do the seismic test. If the result is "positive", then drill for oil; otherwise, do not drill. The ranking payoff is $63,200.

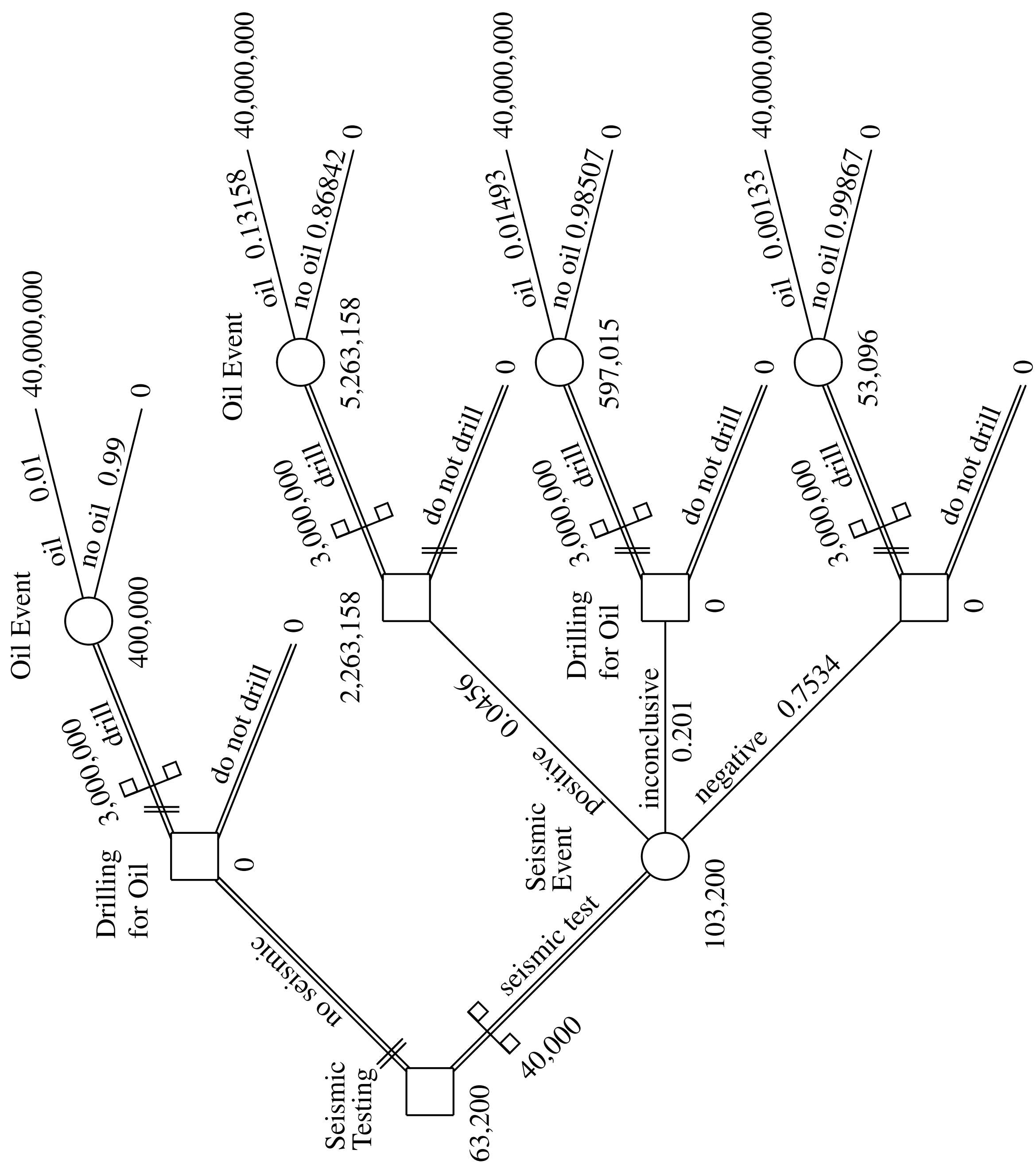


Figure 9.12: Seismic Testing – Rolled-Back Decision Tree

9.4 Decision Making with Sequential Bayesian Revision

9.4.1 Introduction

In this section we look at sequential Bayesian revision. This is used when new information is used to revise the probabilities, and then more new information arrives. This necessitates a second revision of the probabilities. The example is quite long, so it has been analyzed one paragraph at a time. Part A can be solved simply by using a payoff matrix. From this we can find the EVPI, which gives an upper bound to the expected value of any information. We see that the cost of this information is less than the EVPI, so in Part B we proceed with making a decision tree to analyze this situation. This leads to a decision tree and prior and posterior trees which are very similar to those of the oil drilling example of the previous section. Then in Part C we present the concept which is new to this section.

9.4.2 Wood Finishers: Problem Description

Wood Finishers produces a line of executive-type office desks. A high quality desk nets a profit of \$1000. A low quality desk, however, due to refunds and loss of customer goodwill, has a net loss of \$6000. (High or low quality does not refer to the visible part of the desk, which is always of high quality, but rather to the ability to endure years of use.) Ninety-six per cent of the production is of high quality. Adding a rework section to the assembly line would guarantee that each desk would be of high quality, but this would cost \$400 for each desk reworked.

Suppose that the company can inspect each desk at a cost of \$50 (per desk) before deciding whether or not to rework it. The results of the inspection at this station would be one of the following: "looks well," "inconclusive," or "looks poorly." If the inspected desk is of high quality, then there is a 70% chance that the inspection will indicate "looks well," a 20% chance that the inspection will be "inconclusive", and a 10% chance of a "looks poorly" result. If the inspected desk is of low quality then there is a 90% chance of a "looks poorly" result, an 8% chance that the inspection will be "inconclusive", and a 2% chance of "looks well" result.

In addition to the inspection station mentioned above, Wood Finishers can add a second inspection station (which can only inspect a desk which was inspected at the first station). The result of the inspection at the second station is reported as being either "pass" or "fail." If the desk is of high quality there is a 95% chance of a "pass." If the desk is of low quality there is a 97% chance of a "fail." The cost of this test would be $60 per desk inspected. For now, let us suppose that we only need to consider adding the second station if the result of the first test was "inconclusive".

9.4.3 Part A

The first paragraph of the problem description contains a decision (rework) and an event (quality):

> Wood Finishers produces a line of executive-type office desks. A high quality desk nets a profit of $1000. A low quality desk, however, due to refunds and loss of customer goodwill, has a net loss of $6000. (High or low quality does not refer to the visible part of the desk, which is always of high quality, but rather to the ability to last years of use.) Ninety-six per cent of the production is of high quality. Adding a rework section to the assembly line would guarantee that each desk would be of high quality, but this would cost $400 for each desk reworked.

We can analyze this situation with a payoff matrix or a decision tree. The rework decision has two alternatives: do not rework; and rework. If the rework is not done, then there is an event for which there are two possible outcomes: high quality; and low quality.

We do not know how many desks are being made, so we cannot find the absolute level of profit. Instead, we will work out the profit *per desk*.

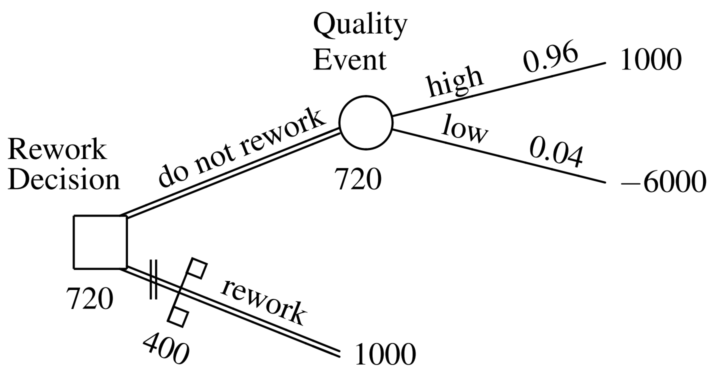
Rework Decision Alternatives	Quality Event Outcomes: High	Low	EV
Do not Rework	1000	−6000	720
Rework	600	600	600
Prob.	0.96	0.04	

Hence, we would choose to not rework, for an expected profit of $720 per desk. The EV with PI is:

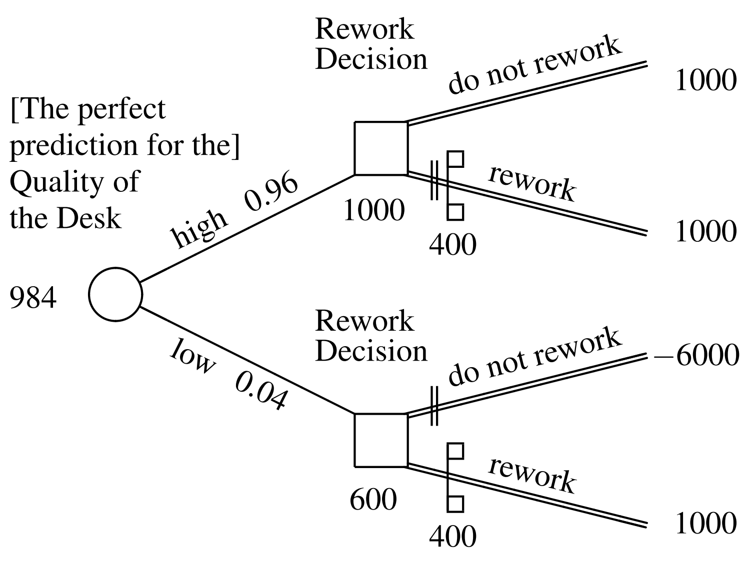
$$\begin{aligned} \text{EV with PI} &= 0.96(1000)+0.04(600) \\ &= 960+24 \\ &= 984 \end{aligned}$$

Hence the EVPI is $984 − $720 = $264 *per desk*.

Although a payoff matrix is perfectly adequate for solving this part of the problem, it is also possible to use a decision tree. Using a tree now helps when drawing the tree for Part B (the second paragraph), because the large tree contains three subtrees which are similar to the one drawn here. Using a tree we obtain:



If we wish to calculate the EV with PI also using a tree, we have:



As before, the EVPI is $984 − $720 = $264 per desk.

9.4.4 Part B

The second paragraph adds an inspection decision and an inspection event:

> Suppose that the company can inspect each desk at a cost of \$50 (per desk) before deciding whether or not to rework it. The results of the inspection at this station would be one of the following: "looks well," "inconclusive," or "looks poorly." If the inspected desk is of high quality, then there is a 70% chance that the inspection will indicate "looks well," a 20% chance that the inspection will be "inconclusive", and a 10% chance of a "looks poorly" result. If the inspected desk is of low quality then there is a 90% chance of a "looks poorly" result, an 8% chance that the inspection will be "inconclusive", and a 2% chance of "looks well" result.

The \$50 cost (per desk) of doing the inspection is much less than the EVPI, which is \$264 (per desk). Hence we must proceed with the analysis to see if it would be worthwhile to do the inspection.

The inspection decision must precede the inspection event, which in turn must precede the main (rework) decision. After the "no inspection" alternative, we are left with the situation which was analyzed in Part A. Therefore, we do not need to redraw this section, but instead merely write the ranking payoff which we calculated to be \$720. For now, we cannot write the probabilities on the inspection outcomes, as these must be determined using Bayesian revision. This part of the tree is shown in Figure 9.13.

After every outcome node we have a sub-tree which resembles the tree made in Part A. Indeed, the only differences are the probabilities, which we need to calculate using Bayesian revision. The tree for Part B without the probabilities is shown in Figure 9.14. We then draw the prior and posterior trees for the Bayesian revision. The completed trees are shown in Figure 9.15. These probabilities are transferred to the decision tree shown in Figure 9.16. Finally, the tree is rolled back to obtain a recommendation. The rolled-back tree is shown in Figure 9.17. Based on this, the recommendation is:

Recommendation Inspect every desk and rework it if and only if a "looks poorly" result is obtained. The ranking payoff is \$869.20 per desk.

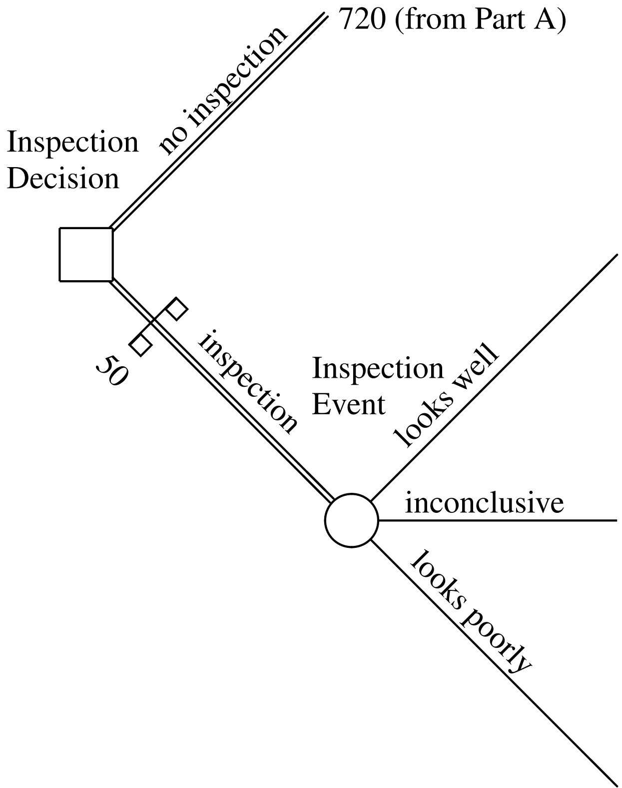


Figure 9.13: Desk Making Decision Tree - Information Decision and Event

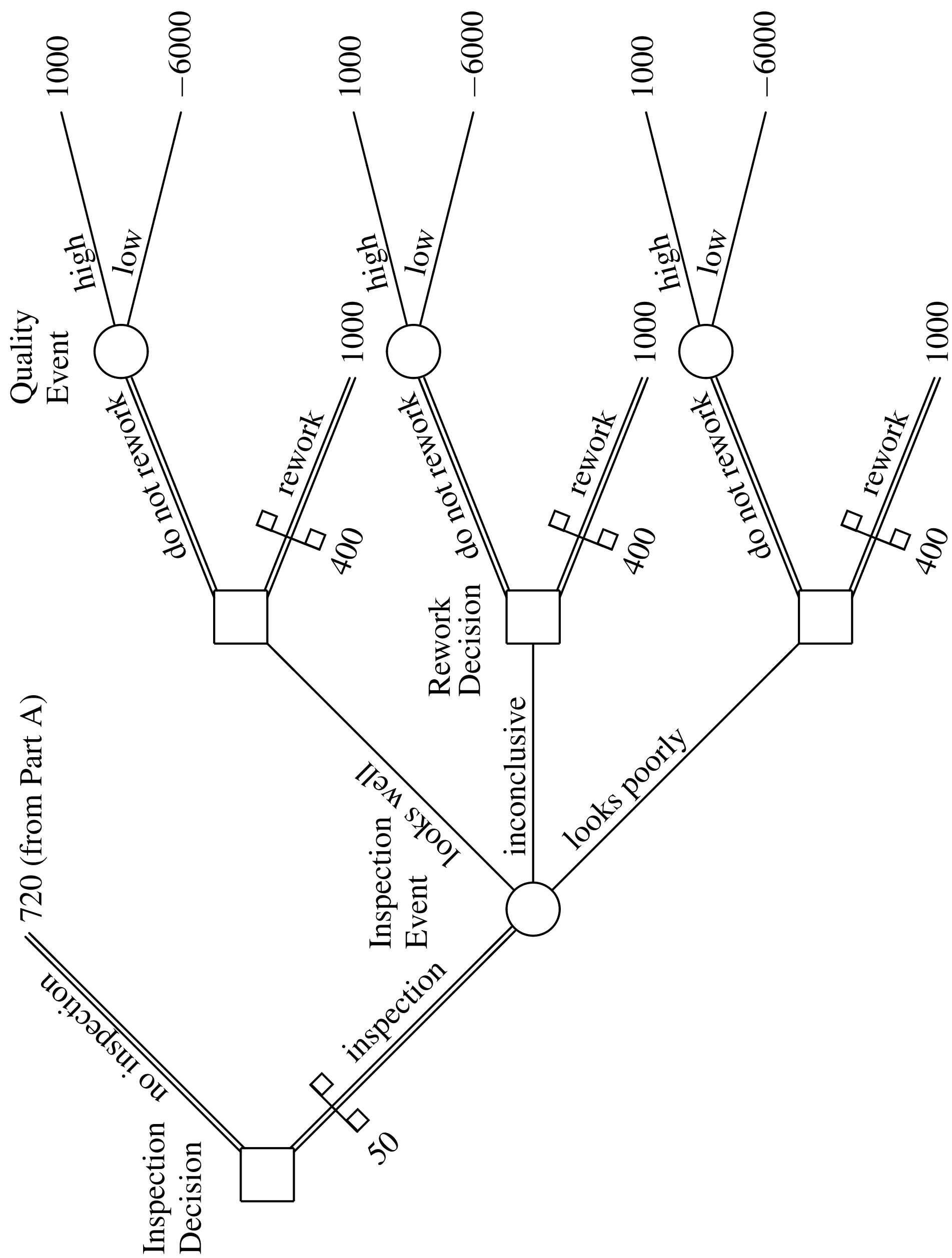


Figure 9.14: Desk Making Decision Tree without Probabilities

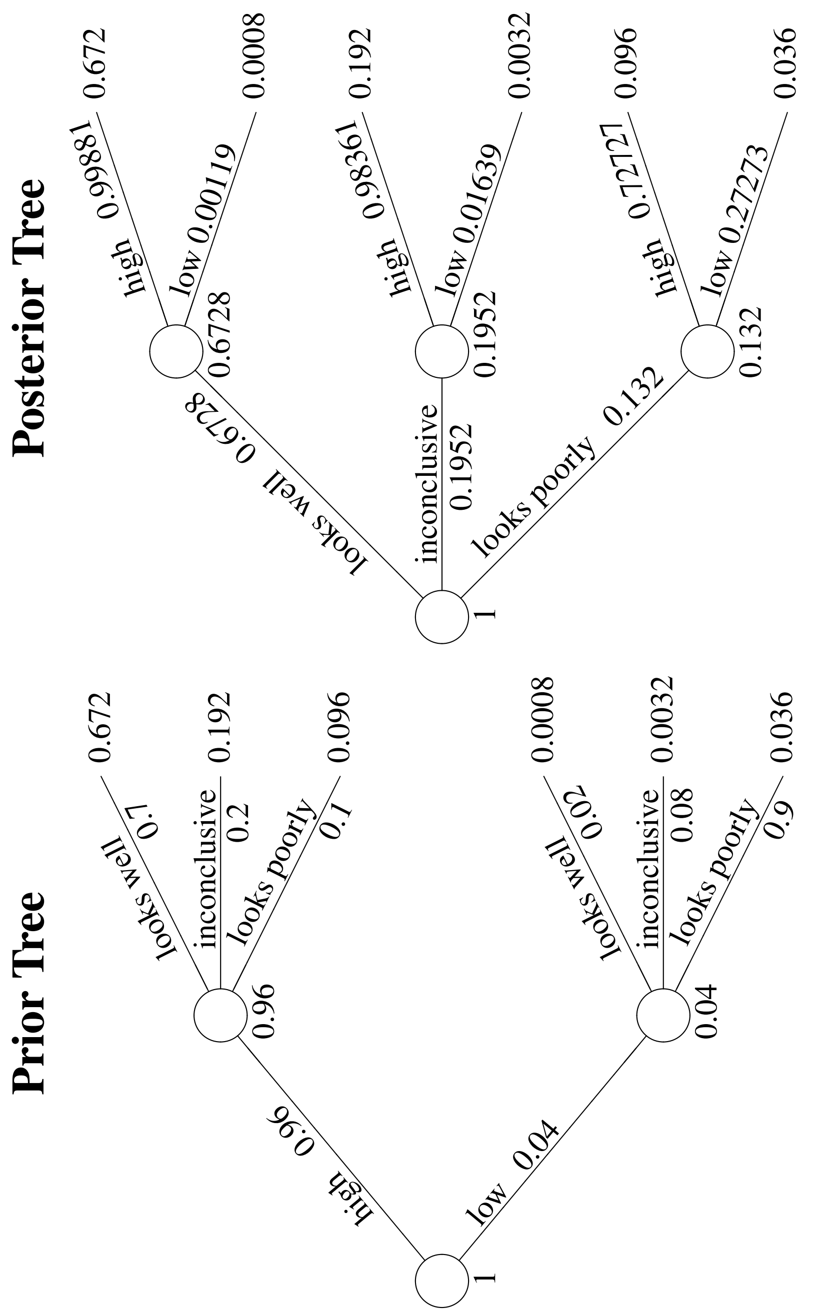


Figure 9.15: Prior and Posterior Trees for the Desk Rework Problem

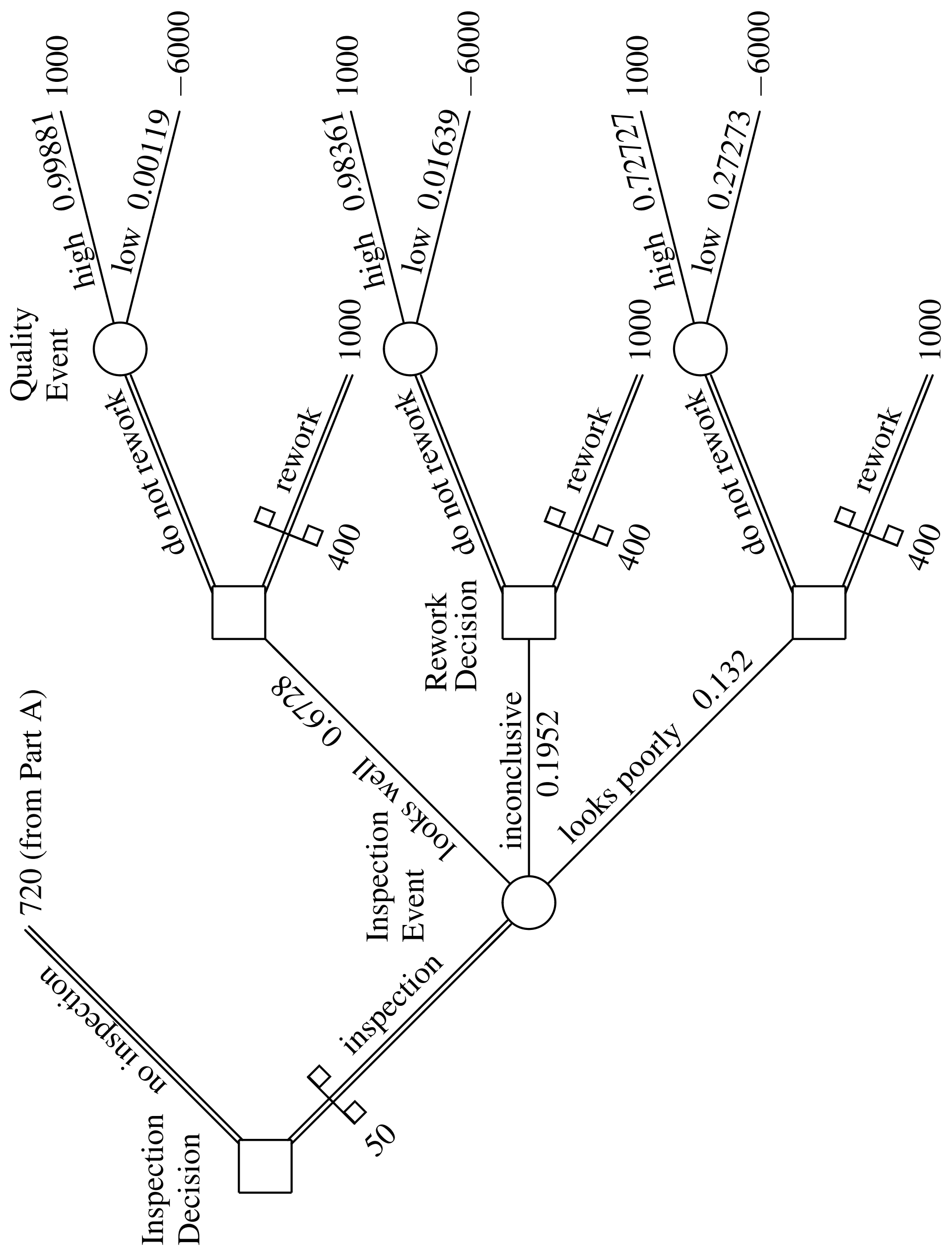


Figure 9.16: Desk Making Decision Tree with Probabilities

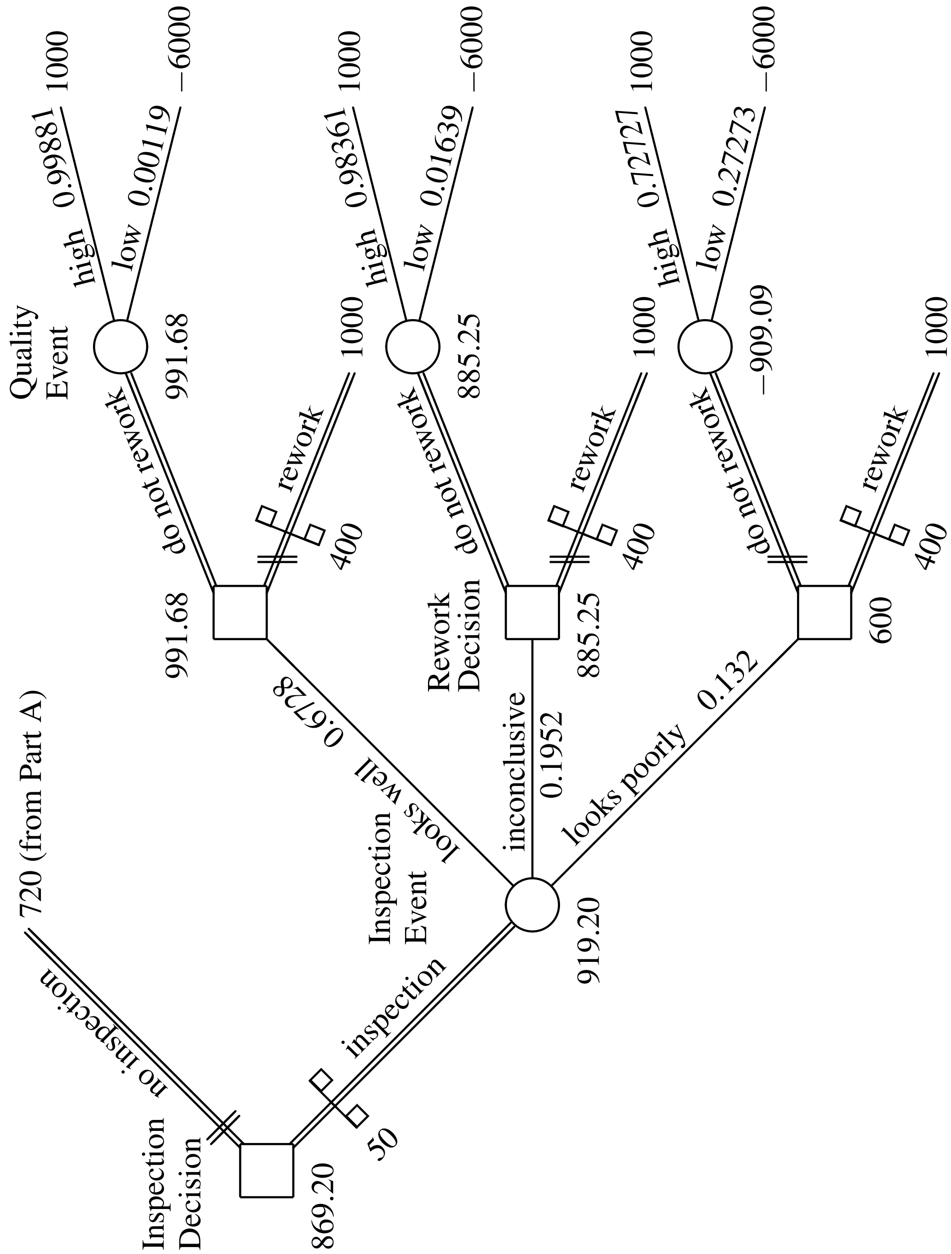


Figure 9.17: Desk Making – Rolled-Back Decision Tree

9.4.5 Part C

Solving Parts A and B required knowing only what we saw in the previous section. However, dealing with the third paragraph requires a bit of thought on how to begin the prior tree:

> In addition to the inspection station mentioned above, Wood Finishers can add a second inspection station (which can only inspect a desk which was inspected at the first station). The result of the inspection at the second station is reported as being either "pass" or "fail." If the desk is of high quality there is a 95% chance of a "pass." If the desk is of low quality there is a 97% chance of a "fail." The cost of this test would be $60 per desk inspected. For now, let us suppose that we only need to consider adding the second station if the result of the first test was "inconclusive".

We will only modify the part of the tree which is affected by this paragraph. We begin with the "inconclusive" outcome branch. After this we can either do or not do the second inspection. If we choose the "no 2nd inspection" alternative, then we are left with the situation which was analyzed in Part B. Therefore, we do not need to redraw this section, but instead merely write the ranking payoff which we calculated to be $885.25.

On the other hand, if we choose to do the second test, then we have an alternative branch with a $60 cost gate, followed by the test event with its two outcomes, "pass", and "fail". For now, we cannot write the probabilities on the inspection outcomes, as these must be determined using Bayesian revision. This part of the tree is shown in Figure 9.18. After both outcome nodes we have a sub-tree which resembles the tree made in Part A. The tree for Part C without the probabilities is shown in Figure 9.19.

The only tricky thing about the Bayesian revision is the determination of the beginning probabilities. The prior tree begins with the "high" and "low" quality outcomes, but the associated probabilities are *not* the 0.96 and 0.04 that we had originally. Instead, we must use the probabilities which are conditional on the first test result being "inconclusive", because they come after the "inconclusive" outcome. Hence we want P(high/inconclusive), which is 0.98361, and P(low/inconclusive), which is 0.01639.

We then draw the prior and posterior trees for the Bayesian revision. The completed trees are shown in Figure 9.20. These probabilities are transferred to the decision tree shown in Figure 9.21. Finally, the tree is rolled back, which

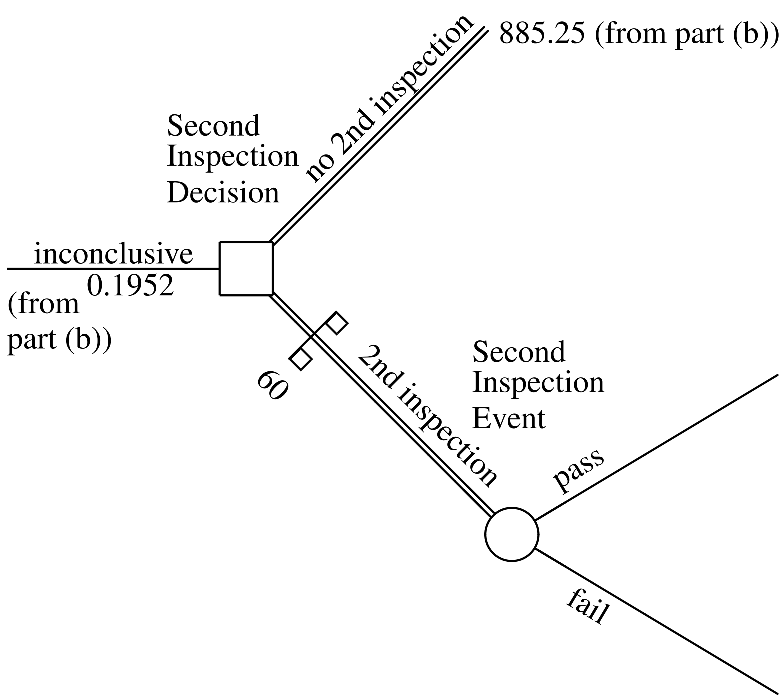


Figure 9.18: Second Test – Beginning of the Decision Tree

is shown in Figure 9.22. Because the payoff after the "inconclusive" branch has increased, the overall recommendation is changed. The increased payoff will cascade through the tree for Part B, increasing the payoff at the outset by:

$$0.1952(910.52 - 885.25) = 4.93$$

Hence the ranking payoff becomes

$$869.20 + 4.93 = 874.13$$

Recommendation Inspect every desk. If a "looks well" result is obtained, then do not rework it. If an "inconclusive" result is obtained, then do the second inspection, and rework it if and only if the result of the second inspection is "fail". If the result of the first inspection is "looks poorly", then rework it. The ranking payoff is $874.13 per desk.

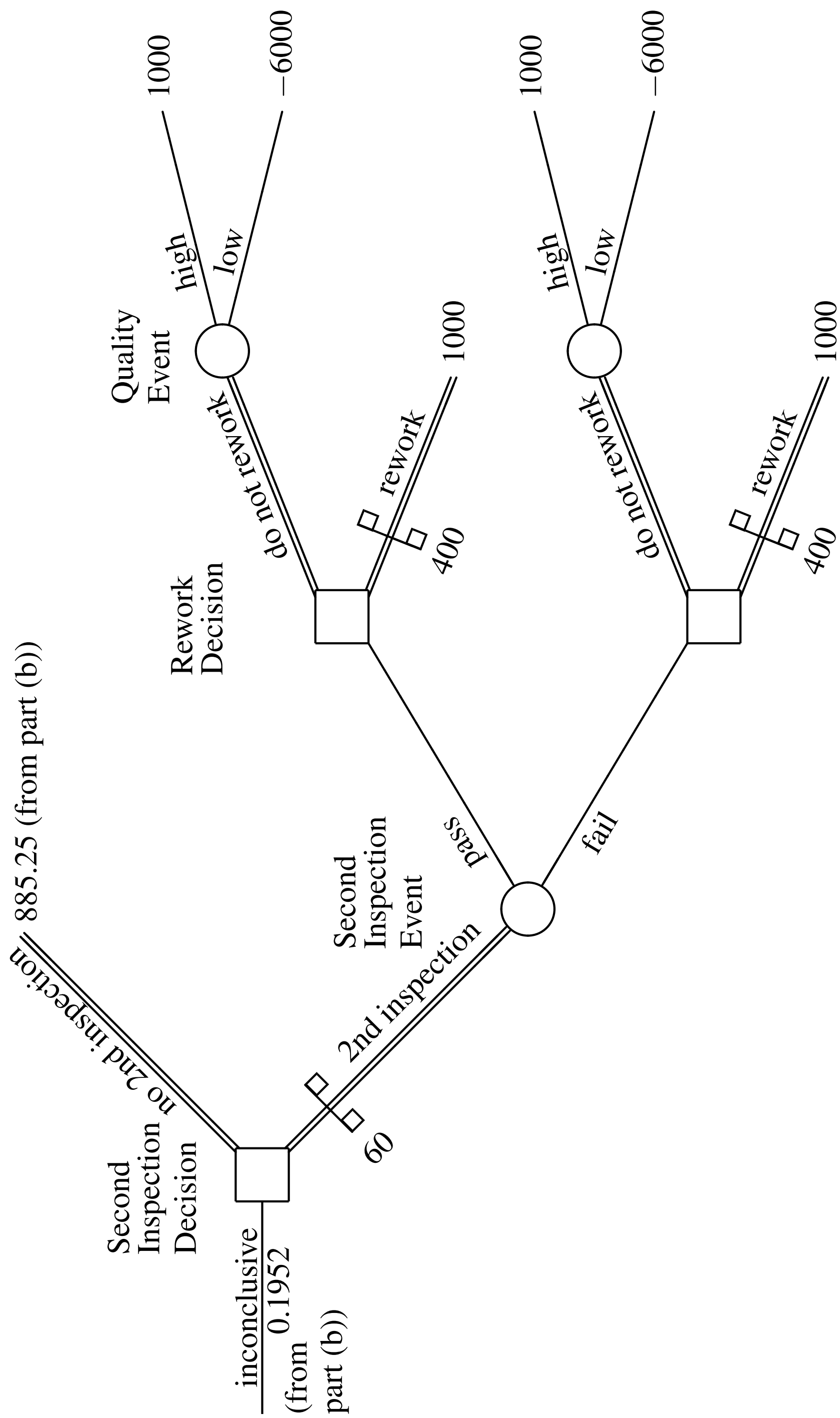


Figure 9.19: Second Test – Decision Tree without Probabilities

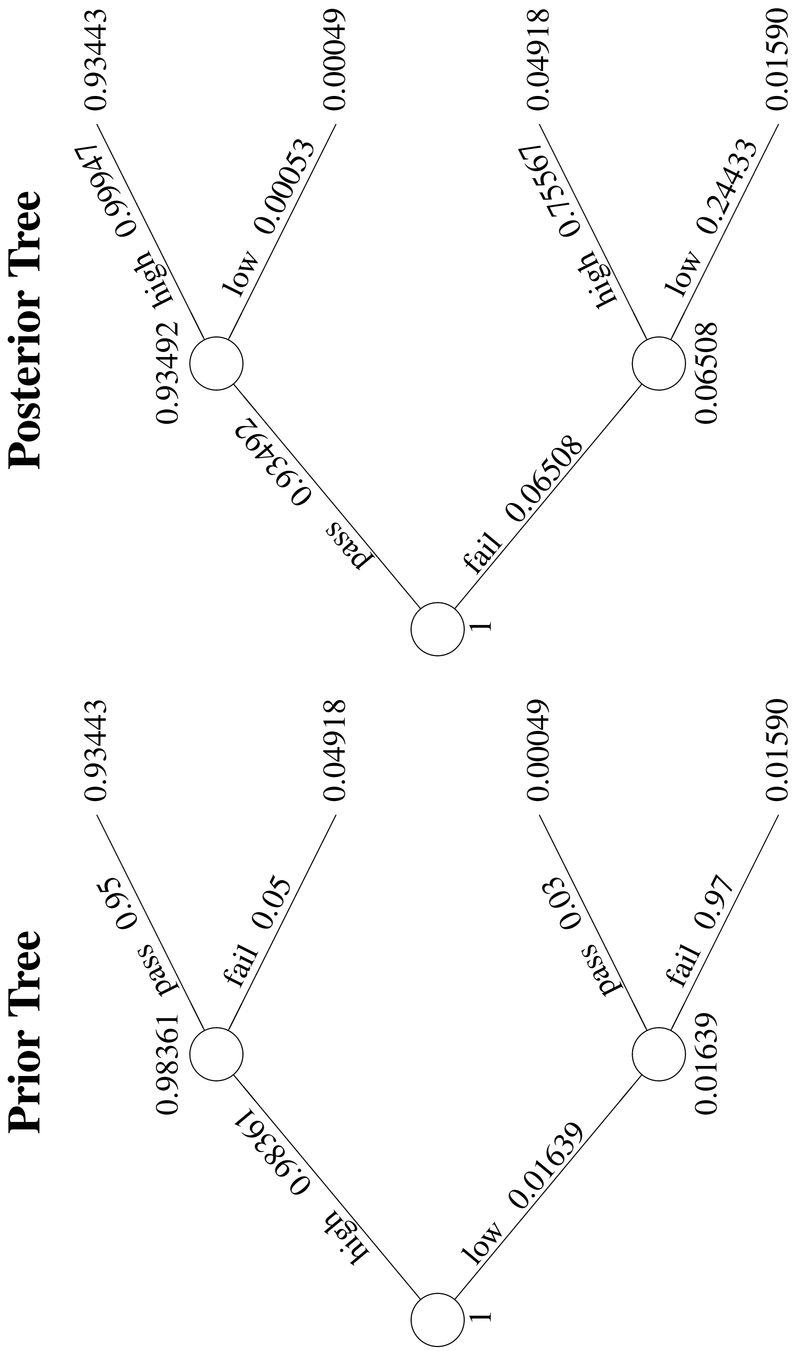


Figure 9.20: Prior and Posterior Trees for the Second Inspection

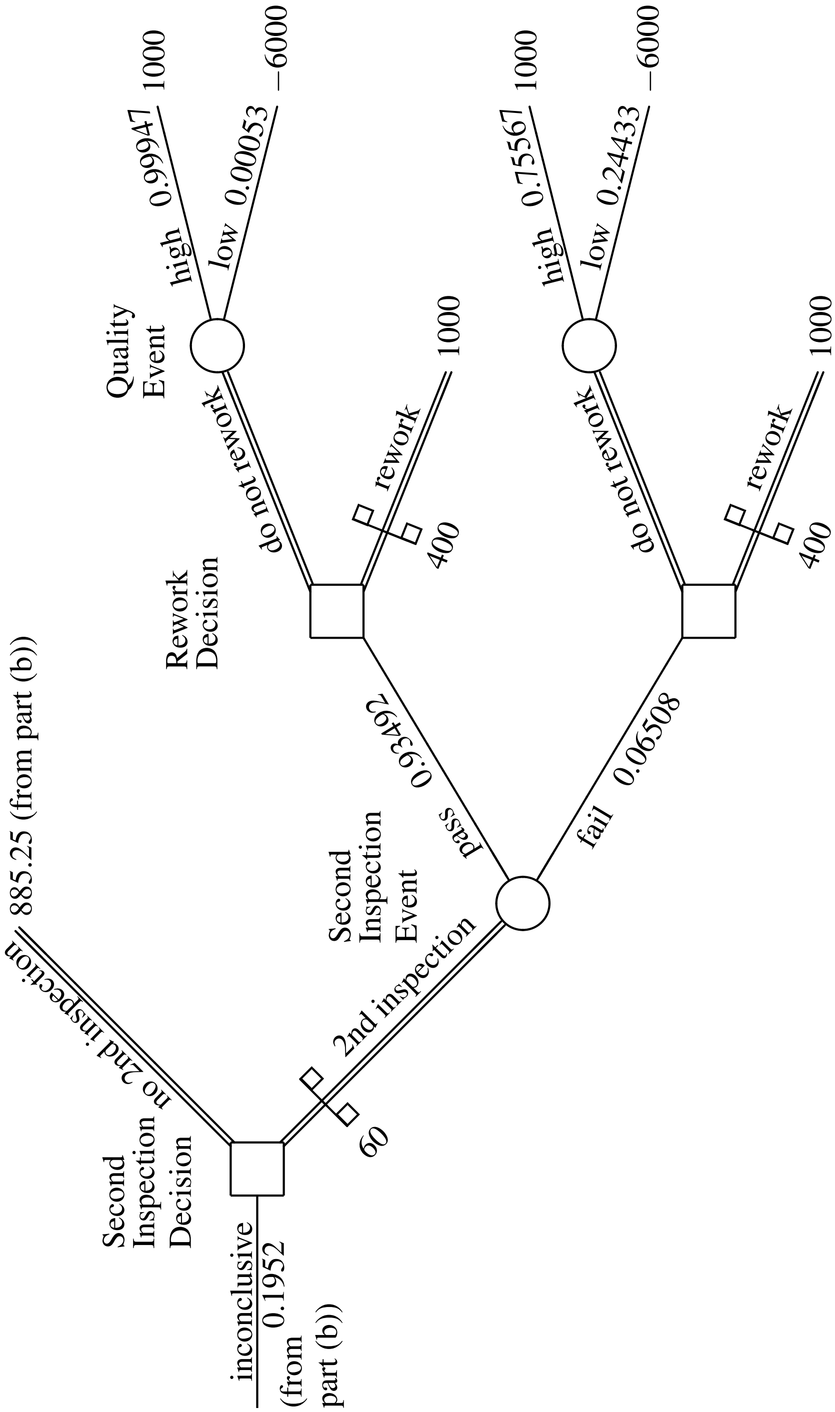


Figure 9.21: Second Test – Decision Tree with Probabilities

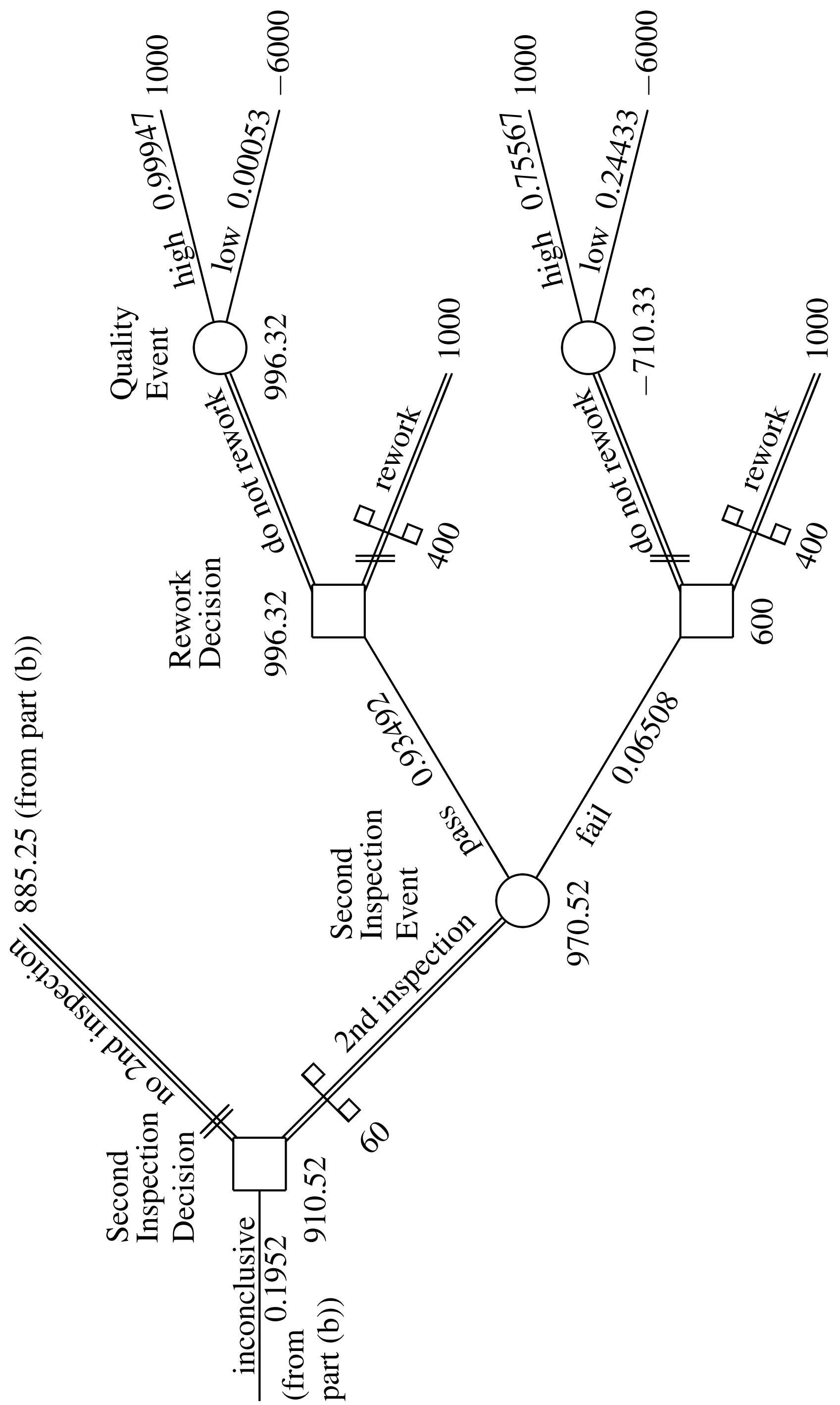


Figure 9.22: Second Test – Rolled-Back Decision Tree

9.5 Problems for Student Completion

9.5.1 Newlab

Newlab has come up with a new product in its research lab. The technical success is clear, but as with any new product the commercial success is risky. Because of this, they would sometimes test-market a product first, and then make a decision about national marketing after the test-market results had come in; at other times they would proceed directly to national marketing. On some occasions, they would abandon the product without even test-marketing it.

The test-marketing would cost about \$120,000. If successful (probability 0.4) there would be revenues of \$40,000; if unsuccessful the revenues would only be \$10,000. Should the test market be successful, a followup national campaign at a cost of \$500,000 would have a 70% chance of success with a revenue of \$1,800,000, otherwise it would be a failure with a revenue of \$150,000. Should the test market be unsuccessful, a followup national campaign would have only a 0.2 chance of success (with the same cost, and the same revenues for success and failure).

A national campaign not preceded by a test campaign would have a 45% chance of success. It would cost \$600,000, and would produce a revenue of \$1,900,000 if successful, but only \$175,000 otherwise.

(a) Draw and solve a decision tree for the situation (using payoff nodes where appropriate), and state the recommendation for Newlab clearly.

(b) If the \$600,000 figure in the last paragraph were changed to \$800,000, what would be the revised recommendation?

9.5.2 Crop Planting

A farmer has been in the habit of always planting potatoes on his farm. In previous years, the seeds for the potatoes were planted in the spring, and were ready to harvest in mid-July. After that, a second planting took place in late July, which was ready to harvest in early October.

This year, however, there is concern that a blight might destroy some or all of the potato crop. One thing he could do would be to plant a different crop such as peas which would not be affected by the blight. The peas would have only a single planting at a cost of \$40,000. This planting would yield a crop in October worth \$70,000 if the weather turns out to be good, or \$30,000 if the weather turns out to be poor. There is a 60% chance that the weather will be good.

If, however, he decides to plant potatoes, he will have to worry about the blight (but the weather has little effect on the potato crop and can be ignored). The potato crop would cost \$60,000 to plant. There is a 10% chance of a severe blight, which would destroy the crop, and render any attempt at a second planting in late July not worth doing. A mild blight (20% chance) would partially destroy the crop, making it worth only \$35,000, while having no blight (70% chance) would produce a crop worth \$80,000. After either a mild blight or no blight, a second planting could be undertaken, with the same costs and revenues as the first. The probability of a severe, mild, or no blight would be 15%, 30%, and 55% if the first planting had a mild blight, but would be 0%, 5%, and 95% if the first planting had no blight.

NOTE: The crop planted in the Spring will be either peas or potatoes; doing a bit of both is not an option in this problem.

Draw the tree, solve it using the rollback procedure, and state the recommendation and the ranking payoff. When drawing the tree, use payoff nodes for intermediate payoffs.

9.5.3 Promising Construction Jobs

John is a self-employed carpenter. For each job John determines the required time if things go according to plan, and a longer time in case there need to be unforseen "extras" (e.g. while replacing clapboard he discovers that insulation is needed). He is paid \$500 per week for each week worked. He knows for sure that he can start either of the two following jobs this coming Monday.

Job/ Client	Time (Weeks) and Probability		Must Begin by the Outset of Week
	No Extras	With Extras	
Bathroom – Mrs. Murphy	3 (0.2)	7 (0.8)	5
Kitchen – Mr. Janes	4 (0.7)	6 (0.3)	6

Each potential customer wants John to do one of three things by Monday: (1) start his/her job, or (2) promise to do his/her job by the required start date, or (3) say that he cannot do the job (the customer would then find another contractor).

If a customer is told at the outset that his/her job cannot begin by the required time, then there's no cost or revenue for that job. If John promises to start a job by the required time, and (i) the promise is kept, he's paid for the actual duration of the job, but (ii) if the promise is not kept, there's no revenue, and there's a cost of \$1000 for lost customer goodwill.

Make and solve a decision tree for this situation.

9.5.4 Consumer Products

A consumer products company has identified a potential product which would require \$1,800,000 in start-up costs to launch. If it turns out to be a major success, there will be a \$10,000,000 contribution to profit. A minor success would give a profit contribution of \$2,000,000, while a failure would have a profit contribution of only \$500,000. The company is most worried about this third possibility, since in this case the net profit would be \$500,000 minus \$1,800,000, i.e. a loss of \$1,300,000. In the past, only one new product in twenty became a major success, while three-quarters of them became failures; there is no reason to suspect that this product would be any different from the rest.

Some of their competitors use an outside independent market research firm to give them advice about new products. The fee for the research firm is \$50,000; in return, the consumer products company would be told that the proposed product either "looks well" or "looks poorly". The research company had established a track record which gave them confidence about saying the following:

1. If a product would be a major success, they would say "looks well" with probability 0.8;
2. If a product would be a minor success, they would say "looks poorly" with probability 0.7;
3. If a product would be a failure, they would say "looks poorly" with probability 0.9.

Develop a decision tree and solve it to obtain a recommendation for the consumer products company.

9.5.5 Desk Rework Problem

Now suppose that in the desk rework problem the second inspection could also be used after a "looks well" or a "looks poorly" outcome from the first test. Determine if the second inspection would be used in either (or both) of these situations, and if so restate the recommendation and the revised ranking payoff at the outset of the tree.

9.5.6 Oil Exploration

It is known that oil exists beneath the surface at a particular location, but it is not known if it's just a small pool of oil (this has a 90% probability), or if there's a large pool of oil. A small pool is worth only \$500,000, but a large pool would be worth \$12 million. To drill (which will determine the size of the pool) would cost \$2,000,000. If they do not drill, an \$80,000 environmental inspection fee will be refunded to them.

A seismic test is available at a cost of \$45,000; the result will be either "positive" or "negative". If a large pool of oil is present then there's an 80% chance of a positive result; if there's just a small pool present then there's a 30% chance of a positive result. There's also a second test available called an EKX test. This could only be used after doing a seismic test and obtaining a positive result, and if used would cost \$40,000. If there's a large pool of oil the EKX test will report "high" with probability 0.85; if there's a small pool the EKX test will report "low" with probability 0.95.

The company has decided that if they do a seismic test and if it turns out to be negative, or if after a positive seismic test they do an EKX test and it turns out to be low, then they will not drill.

Draw and solve a decision tree to determine a recommendation for this situation. Please use five decimal place accuracy for the Bayesian revisions.

9.6 More Difficult Problems

As these problems might be used for hand-in assignments, solutions are not provided.

9.6.1 Payoff Matrix with Binomial Demand

Demand for the *Telegram* at the Avalon News Depot can range anywhere from 31 to 49 papers per day, according to the following binomial probability distribution:

$$P(\mathbf{K}=k)=\frac{18!}{k!(18-k)!}0.3^k(1-0.3)^{18-k}$$

where **K** takes on the values of the number of newspapers demanded in *excess* of 31, ranging from 0 to 18 inclusive. For example, to find the probability that 36 papers will be ordered we use $k=5$ in the formula to obtain 0.201725.

The Avalon News Depot sells the papers for \$1.50 each. Their buying price depends on the quantity ordered:

Quantity (per day)	Buying Price (per paper)
40 or fewer	\$1.30
41 to 46 inconclusive	\$1.17
47 or more	\$1.03

Papers that are not sold by the end of the day are sold to a recycling firm for 5 cents each.

They wish to know how many papers they should order so that their expected profit is maximized.

Using the BINOMDIST function to create the probability row, create a spreadsheet model for this situation, determining the expected profit for each possible order quantity from 31 to 49 inclusive. Using the ability of Excel to make charts (also called graphs), plot expected profit as a function of the (integer) quantity ordered. State the optimal number of papers to order.

9.6.2 Future Shock

Future Shock is a retail chain specializing in electronic equipment. It is currently having some problems with one of its high volume items. These items are ordered from the supplier in lots of 100 units and give a profit contribution of \$50 per unit. Past experience indicates that the possible percentage defective in a lot are 10%, 20% and 30% with probabilities of 0.5, 0.3 and 0.2 respectively. Future Shock could non-destructively test some or all of the units. A maximum of three units could be tested sequentially (i.e. test one and see whether its defective or not defective, then test another, and so on); however, at any time they may choose to simultaneously test all remaining units. (For example, after testing two units, they could choose to do no more testing, or test the third unit, or test all remaining 98 units.) No matter how the testing is done, the testing costs \$20 per unit tested. If Future Shock inspects a unit and finds it to be defective, then it will be replaced by the supplier at no cost. The replacement unit will be known to be non-defective. However, when a defective unit is sent to a customer, it is replaced by Future Shock at a cost of \$100. Determine your recommendation for Future Shock, doing all calculations on a spreadsheet.

Index

Appendix A

Software for Optimization

As an alternative to using a spreadsheet solver, one can use software which has been created specifically for linear/integer/nonlinear programming. First we look at LINGO, and then briefly discuss other programs.

A.1 LINGO

A.1.1 How to Obtain It

LINGO is a registered trademark of LINDO Systems Inc. The company's first product was named LINDO, and that name has been retained in the name of the company. Though LINDO is still available, development on this software ended in 2003, and no version was ever created for the Apple Macintosh. By contrast, version 18 of the much more advanced LINGO was released in 2018, and variants exist for Windows, Mac, and Linux.

The website for LINDO Systems Inc. is at https://www.lindo.com/. To obtain LINGO, go to the website, then click on "Downloads", then scroll down and click on "Download LINGO". A page will appear with many versions of LINGO; choose one to download, and then open the zip file.

The software can be operated in "Demo Version" mode with restrictions on problem size. As a student, working on non-commercial research, it may be possible to obtain an access code for an unlimited version, provided by the generosity of LINDO Systems Inc.

A complete list of the features on LINGO can be obtained by downloading the very extensive LINGO User's Manual from https://lindo.com/index.php/lsdownloads/user-manuals. A few features to help the user get started are described below.

A.1.2 Introduction to Solving Linear Models

LINGO has two ways of being used. One way, which is suitable for smaller models and is more easily understood by beginners, inputs each line in a manner which is similar to the algebraic model. We will illustrate this simpler approach using four examples. There is also a more complicated approach which is suitable for larger models, which uses sets to separate the data from the variables. This approach is beyond the scope of this book.

We are not required to enter the variable definitions, because they are not needed to solve the problem mathematically. However, we may wish to enter them as comments in order to make things easier to understand. A comment is made by first typing an exclamation mark; a comment is ended by typing a semicolon. Anything from the exclamation mark to the semicolon inclusive is ignored by LINGO. Comments could also be made to give the name of the model, the name of the person who made it, the date of its creation, the purpose of the constraints, or anything else that might make the file easier to understand when viewing it at a later point in time. Also, blank lines may be inserted at will to help improve the appearance of the file.

We do not enter the non-negativity restrictions, because they are always assumed to be present. Some adjustments have to be made because of the limitations of the keyboard. We cannot enter a subscripted variable, hence X_1 and X_2 are entered as `X1` and `X2`. Also, since there are no $\leq$ or $\geq$ symbols on the keyboard, we enter <= and >= instead. To give the purpose of a constraint or set of constraints, we input a comment line.

Comments in LINGO will appear in green, commands are in blue, and everything else is in black. There is a great deal of similarity with the original algebraic model, but here are some important exceptions:

1. The objective function, and each constraint, ends with a semicolon.

2. The word "maximize" is invoked with "MAX ="; and we use "MIN =" for minimization. LINGO will write MAX or MIN in blue.

3. Multiplication requires an asterisk. Hence "$4X_1$" is entered as "$4*X1$".

4. The words "subject to" are not entered.

5. The non-negativity restrictions are assumed; they are not entered by the user.

6. We finish with the command “END” (which LINGO will write in blue).

After doing the above for any model we:

1. Save the model by clicking on **File**, and under this clicking in either **Save** or **Save As...**. The default extension is lg4.

2. Click on **Solver**, and then under this click on **Solve**. The Solution Report will open in a second window.

A.1.3 Solving the Cement Plant Problem

Converting the Cement Algebraic Model to LINGO Syntax

To illustrate the use of LINGO for a linear optimization model, we will use the algebraic model of the cement problem, which appears on page 44, and is repeated here.

$$
\begin{aligned}
X_1 &= \text{ the number of TPD of Type 1 cement made} \\
X_2 &= \text{ the number of TPD of Type 2 cement made}
\end{aligned}
$$

$$
\begin{array}{rrcrcr}
\text{maximize} & 8X_1 & + & 10X_2 & & \\
\text{subject to} & & & & & \\
\text{Type 1 Sales} & X_1 & & & \geq & 40 \\
\text{Type 2 Sales} & & & X_2 & \geq & 30 \\
\text{Total Production} & X_1 & + & X_2 & \leq & 200 \\
\text{Dept. A Labour} & 3X_1 & + & 2X_2 & \leq & 585 \\
\text{Dept. B Labour} & 1.5X_1 & + & 5X_2 & \leq & 500 \\
\text{Dept. C Labour} & 4X_1 & + & 6X_2 & \leq & 900 \\
\text{non-negativity} & X_1 & , & X_2 & \geq & 0
\end{array}
$$

This is entered into LINGO (the colours are made by LINGO) as:

```
! Cement Plant Model
X1 = the number of TPD of Type 1 cement made
X2 = the number of TPD of Type 2 cement made;
MAX = 8 * X1 + 10 * X2;
! Sales;
X1 >= 40;
X2 >= 30;
! Total Production;
X1 + X2 <= 200;
! Labour in Departments A, B, and C;
3 * X1 + 2 * X2 <= 585;
1.5 * X1 + 5 * X2 <= 500;
4 * X1 + 6 * X2 <= 900;
END
```

This model can be saved by clicking on **File**, and under this clicking in either **Save** or **Save As...**. The default extension is lg4. Hence if the prefix is *cement*, the file name will be *cement.lg4*.

Now click on **Solver**, and then under this click on **Solve**. The following report is obtained:

```
Global optimal solution found.
  Objective value:                              1700.000
  Infeasibilities:                              0.000000
  Total solver iterations:                             4
  Elapsed runtime seconds:                          0.05

  Model Class:                                        LP

  Total variables:                    2
  Nonlinear variables:                0
  Integer variables:                  0

  Total constraints:                  7
  Nonlinear constraints:              0

  Total nonzeros:                    12
  Nonlinear nonzeros:                 0


                        Variable           Value        Reduced Cost
                              X1        150.0000            0.000000
                              X2        50.00000            0.000000

                             Row    Slack or Surplus      Dual Price
                               1        1700.000            1.000000
                               2        110.0000            0.000000
                               3        20.00000            0.000000
                               4        0.000000            4.000000
                               5        35.00000            0.000000
                               6        25.00000            0.000000
                               7        0.000000            1.000000
```

We see the objective function value of 1700 on the second line. The next several lines give us a measure of how much work the computer did to find the optimal solution – this is of technical rather than managerial interest, and we shall not use this information.

Then we see that the values of the variables are $X_1 = 150$ and $X_2 = 50$. It is up to the user of the software to translate this into the words needed to express a recommendation.

Note that LINGO labels the objective function as row 1, the first constraint as row 2, and so on, with the sixth constraint being row 7.

Sensitivity Analysis

The subject of sensitivity analysis is covered in Chapter 4. The term *Slack or Surplus* is explained in Chapter 2 and then again in Chapter 4. The terms *Reduced*

Cost and *Dual Prices* (see the footnote on page 168) are explained in Chapter 4. Further use of sensitivity analysis requires the allowable ranges. To obtain these using LINGO we need to do the following:

(a) Under **Solver**, click on **Options**.

(b) Near the top of the dialog box, click on the **General Solver** tab.

(c) Find the **Dual Computations** box, and set it to *Prices and Ranges*.

(d) To make this change permanent, at the bottom click on **Apply** and then **Save**.

Now run the model by clicking on **Solver** and then **Solve**, which creates the Solution Report. Then, close the Solution Report window and click on **Solver** and under this click on **Range**. Now, the Range Report will open. Here is the Range Report for the Cement problem:

```
Ranges in which the basis is unchanged:

                              Objective Coefficient Ranges:

                            Current        Allowable        Allowable
          Variable      Coefficient         Increase         Decrease
                X1         8.000000         2.000000         1.333333
                X2         10.00000         2.000000         2.000000

                                 Righthand Side Ranges:

                            Current        Allowable        Allowable
               Row              RHS         Increase         Decrease
                 2         40.00000         110.0000         INFINITY
                 3         30.00000         20.00000         INFINITY
                 4         200.0000         7.000000         4.545455
                 5         585.0000         INFINITY         35.00000
                 6         500.0000         INFINITY         25.00000
                 7         900.0000         14.28571         40.00000
```

Now we present some more examples of models put into LINGO syntax. They include a minimization model, a larger maximization model with double-subscription variables, and a model with integer variables.

A.1.4 The Diet Model in LINGO Syntax

The algebraic model of the diet problem seen in Chapter 2 appears on page 64, and is repeated here.

X_1 = the number of double hamburgers eaten each day
X_2 = the number of servings of orange juice drunk each day

$$
\begin{array}{rrcrcr}
\text{minimize} & 1.25X_1 & + & 0.32X_2 & & \\
\text{subject to} & & & & & \\
\text{Protein RDI} & 31.820X_1 & + & 1.469X_2 & \geq & 46 \\
\text{Iron RDI} & 5.547X_1 & + & 1.096X_2 & \geq & 15 \\
\text{Vitamin C RDI} & 1.075X_1 & + & 85.656X_2 & \geq & 60 \\
\text{Iron Proportion} & 0.5547X_1 & - & 0.9864X_2 & \leq & 0 \\
\text{non-negativity} & X_1 & , & X_2 & \geq & 0
\end{array}
$$

```
! Diet Model
X1 = the number of double hamburgers eaten each day
X2 = the number of servings of orange juice drunk each day;
MIN = 1.25 * X1 + 0.32 * X2;
! RDI for Protein, Iron, and Vitamin C;
31.820 * X1 + 1.469 * X2 >= 46;
5.547 * X1 + 1.096 * X2 >= 15;
1.075 * X1+ 85.656 * X2 >= 60;
! Iron Proportion;
0.5547 * X1 - 0.9864 * X2 <= 0;
END
```

A.1.5 The Gasoline Blending Model in LINGO Syntax

The model for the gasoline blending problem appears on page 93, and is repeated here.

X_1 = amount (in m^3) of output gasoline #1 sold,
X_2 = amount (in m^3) of output gasoline #2 sold,
I_1 = amount (in m^3) of input gasoline #1 purchased,
I_2 = amount (in m^3) of input gasoline #2 purchased,

$U_{1,1}$ = amount (in m^3) of input 1 used to make output 1,

$U_{1,2}$ = amount (in m^3) of input 1 used to make output 2,

$U_{2,1}$ = amount (in m^3) of input 2 used to make output 1,

$U_{2,2}$ = amount (in m^3) of input 2 used to make output 2.

$$
\begin{array}{rrcl}
\text{maximize} & 310X_1 + 230X_2 - 265I_1 - 188I_2 & & \\
\text{subject to} & & & \\
\text{Available, Input 1} & I_1 & \leq & 25000 \\
\text{Available, Input 2} & I_2 & \leq & 60000 \\
\text{Minimum production, Output 1} & X_1 & \geq & 15000 \\
\text{Minimum production, Output 2} & X_2 & \geq & 30000 \\
\text{Balance, Input 1} & -I_1 + U_{1,1} + U_{1,2} & = & 0 \\
\text{Balance, Input 2} & -I_2 + U_{2,1} + U_{2,2} & = & 0 \\
\text{Balance, Output 1} & -X_1 + U_{1,1} + U_{2,1} & = & 0 \\
\text{Balance, Output 2} & -X_2 + U_{1,2} + U_{2,2} & = & 0 \\
\text{Octane Rating, Output 1} & -95X_1 + 110U_{1,1} + 80U_{2,1} & \geq & 0 \\
\text{Octane Rating, Output 2} & -85X_2 + 110U_{1,2} + 80U_{2,2} & \geq & 0 \\
\text{Vapour Pressure, Output 1} & -40X_1 + 35U_{1,1} + 65U_{2,1} & \leq & 0 \\
\text{Vapour Pressure, Output 2} & -55X_2 + 35U_{1,2} + 65U_{2,2} & \leq & 0 \\
 & & & \\
 & \text{all variables must be} & \geq & 0
\end{array}
$$

In LINGO, variable names cannot contain commas. In this small example we can simply omit the commas in the U variables. Hence, for example, $U_{1,2}$ is entered as U12.[1] Hence the model is entered into LINGO as:

[1] In a more complicated model, with variables $X_{3,27}$ and $X_{32,7}$ we could not simply eliminate the commas because this would make both variables X327. However, we could change each comma to say a C, making the variables in LINGO X3C27 and X32C7.

```
! Gasoline Blending Model
All quantities are in cubic metres.
X1 = amount of output gasoline #1 sold,
X2 = amount of output gasoline #2 sold,
I1 = amount of input gasoline #1 purchased,
I2 = amount of input gasoline #2 purchased,
U11 = amount of input 1 used to make output 1,
U12 = amount of input 1 used to make output 2,
U21 = amount of input 2 used to make output 1,
U22 = amount of input 2 used to make output 2;
MAX = 310 * X1 + 230 * X2 - 265 * I1 - 188 * I2;
! Inputs;
I1 <= 25000;
I2 <=  60000;
! Outputs;
X1 >= 15000;
X2 >= 30000;
! Balances;
- I1 + U11 + U12 = 0;
- I2 + U21 + U22 = 0;
- X1 + U11 + U21 = 0;
- X2 + U12 + U22 = 0;
! Octane Rating;
- 95 * X1 + 110 * U11 + 80 * U21 >= 0;
- 85 * X2 + 110 * U12 + 80 * U22 >= 0;
! Vapour Pressure;
- 40 * X1 + 35 * U11 + 65 * U21 <= 0;
- 55 * X2 + 35 * U12 + 65 * U22 <= 0;
END
```

A.1.6 The Cargo Plane Model in LINGO Syntax

Recall that Cargo Plane Model, first seen as a spreadsheet model in Chapter 1, and then formulated as an algebraic model in Chapter 6 on page 242, is as follows:

$$X_1 = \text{the number of Type 1 boxes carried on the plane}$$
$$X_2 = \text{the number of Type 2 boxes carried on the plane}$$

$$
\begin{array}{rcccccc}
\text{One of:} & & & & & & \\
\text{(i) maximize} & 400X_1 & + & 400X_2 & & & \\
\text{(ii) maximize} & 600X_1 & + & 250X_2 & & & \\
\text{(iii) maximize} & 300X_1 & + & 750X_2 & & & \\
\text{subject to} & & & & & & \\
\text{Volume} & 2.9X_1 & + & 1.8X_2 & \leq & 15 \\
\text{Mass} & 470X_1 & + & 530X_2 & \leq & 3600 \\
\text{Type 1} & X_1 & & & \leq & 6 \\
\text{Type 2} & & & X_2 & \leq & 8 \\
\text{non-negativity} & X_1 & , & X_2 & \geq & 0 \\
\text{integer} & X_1 & , & X_2 & &
\end{array}
$$

We need to give some functions needed for defining integer variables:

0/1 Integer Variables When a variable has to be either 0 or 1, we use the @BIN function to make this declaration. This command could be placed anywhere; it could go before the MAX command, or it could be placed just before the END command. If say variable Y_3 has to be either 0 or 1, then somewhere we would write:

@BIN(Y3);

General Integer Variables When a variable has to be a positive integer, we use the @GIN function to make this declaration. If say variable X_5 has to be integer, then somewhere we would write:

@GIN(X5);

For this example, we need to define two general integer variables. Using version (i) of the objective function the model in LINGO's syntax this is:

```
! Cargo Plane Model
X1 = the number of Type 1 boxes carried on the plane
X2 = the number of Type 2 boxes carried on the plane;
MAX =   400 * X1 + 400 * X2;
! Volume;
2.9 * X1 + 1.8 * X2 <= 15;
! Mass;
470 * X1 + 530 * X2 <= 3600;
! Type 1;
X1 <= 6;
! Type 2;
X2 <= 8;
@GIN(X1);
@GIN(X2);
END
```

A.2 Other Software

Many companies make software for linear programming. Of these, most offer either a no-cost or at least a cheap version for students. There is also some open-source software. Here is some of what is available:

1. GLPK is open-source software. See http://www.gnu.org/software/glpk/.

2. COIN-OR is open-source software. See https://www.coin-or.org/. It is an umbrella for many projects, such as CMPL at http://www.coliop.org/.

3. CPLEX, an IBM product, can handle very large-scale examples. IBM makes the program free in two versions. Firstly, there is a "Community Edition", which can handle up to 1000 variables and 1000 constraints; this version may be downloaded by anyone from https://www.ibm.com/account/reg/us-en/signup?formid=urx-20028. Secondly, as part of their "Academic Initiative", an unrestricted version is available for academic non-commercial use. More information is available at https://ibm.onthehub.com/WebStore/ProductSearchOfferingList.aspx?srch=ilog+cplex.

4. Gurobi has a website for users at universities at http://www.gurobi.com/academia/for-universities.

5. AMPL is not itself a solver for linear optimization, but instead is a modeling system for large-scale applications. AMPL can be linked to a wide variety of solvers, such as CPLEX and Gurobi. See https://ampl.com/.

Appendix B

Dedicated Network Algorithms

Here we describe two purpose-built algorithms, on for the maximal flow problem, and the other for the shortest path problem.

B.1 Maximal Flow Algorithm

Here we describe a purpose-built algorithm for solving the maximal flow problem. To discuss this problem properly, we need to define three terms: cut, cut capacity, and minimal cut capacity.

B.1.1 Definition of Terms

A *cut* partitions the nodes into two connected groups of nodes, with the source in one group, and the sink in the other. These groups can be formed by drawing a line through the network which "cuts" the network. The cut consists of each arc which *directly* connects the nodes in the source group with those in the sink group. Two examples are:

	Source Group	Sink Group	Cut Arcs
(a)	$\boxed{1}$, $\boxed{3}$	$\boxed{2}$, $\boxed{4}$, $\boxed{5}$, $\boxed{6}$	(1,2), (3,2), and (3,5)
(b)	$\boxed{1}$, $\boxed{2}$, $\boxed{4}$	$\boxed{3}$, $\boxed{5}$, $\boxed{6}$	(1,3), (2,3), (4,5), and (4,6)

The cut which forms the two groups in (a) is illustrated in Figure B.1. While the cut does not have to be represented by a straight line, as it is here, the cut cannot loop back across the network. For example, two groups such as $\boxed{1}$,$\boxed{2}$,$\boxed{5}$ and $\boxed{3}$,$\boxed{4}$,$\boxed{6}$ would not be formed from a cut.

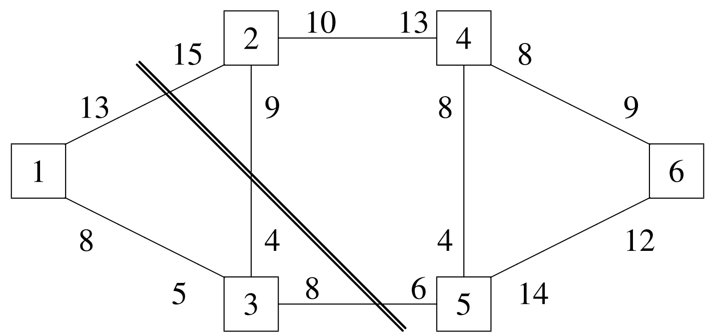


Figure B.1: A cut through arcs (1,2), (3,2), and (3,5)

The *cut capacity* is the sum of the capacities (in the source to sink direction) of all the arcs which cross the "cut."

For the previous examples we obtain:

	Cut Arcs	Cut Capacity
(a)	(1,2), (3,2), (3,5)	13 + 4 + 8 = 25
(b)	(1,3), (2,3), (4,5), (4,6)	8 + 9 + 8 + 8 = 33

For a small problem such as this one, we can enumerate all of the cuts and their associated capacities as follows:

Cut #	Source Group	Sink Group	Cut Arcs	Cut Capacity
1	1	2, 3, 4, 5, 6	(1,2), (1,3)	13 + 8 = 21
2	1, 3	2, 4, 5, 6	(1,2), (3,2), (3,5)	13 + 4 + 8 = 25
3	1, 3, 5	2, 4, 6	(1,2), (3,2), (5,4), (5,6)	13 + 4 + 4 + 14 = 35
4	1, 2	3, 4, 5, 6	(1,3), (2,3), (2,4)	8 + 9 + 10 = 27
5	1, 2, 3	4, 5, 6	(2,4), (3,5)	10 + 8 = 18
6	1, 2, 3, 5	4, 6	(2,4), (5,4), (5,6)	10 + 4 + 14 = 28
7	1, 2, 4	3, 5, 6	(1,3), (2,3), (4,5), (4,6)	8 + 9 + 8 + 8 = 33
8	1, 2, 3, 4	5, 6	(3,5), (4,5), (4,6)	8 + 8 + 8 = 24
9	1, 2, 3, 4, 5	6	(4,6), (5,6)	8 + 14 = 22

The *minimal cut capacity* of a network with a given source and sink is the smallest cut capacity which is obtained when every possible cut has been examined. In this example, the minimal cut capacity is 18 (cut # 5).

There is a theorem called the *Max Flow/Min Cut Theorem* which states that the maximal flow from source to sink is equal to the minimal cut capacity. Thus, the maximal flow from $\boxed{1}$ to $\boxed{6}$ in this example is 18. If either the source or the sink were to change, we would have a different set of cuts, and would have to calculate the cut capacities from scratch. We could use this theorem to find the maximal flow between any pair of nodes, however, the max flow/min cut theorem is rarely used on its own at the outset. Instead, there is an efficient algorithm for this problem, which finds not only the value of the maximal flow, but also determines the flow on each arc from source to sink. The use of the theorem comes at the end of the algorithm, when it is used merely to prove that the solution is optimal.

The algorithm begins with no flow from source to sink. At each iteration the flow from source to sink is augmented, and the arc capacities are adjusted. When no further augmentation to the flow is possible, the algorithm stops.

B.1.2 Maximal Flow Problem Algorithm

Step 0: Flow = 0

Step 1: Find any path from the source to the sink for which all the forward arc capacities are strictly positive (i.e. > 0). If no such path exists, then the optimal solution has been found, with the maximal flow being the current value of "Flow."

Step 2: Let $C_{\min}$ be the smallest capacity on the path found in Step 1. Increase the flow from the source to the sink by sending (an additional) $C_{\min}$ units of flow over this path.

Flow $\leftarrow$ Flow + $C_{\min}$.[1]

Step 3: Adjust for the increase in flow as follows:

(i) *decrease* all arc capacities along this path in the forward (i.e. source to sink) direction by $C_{\min}$.

(ii) *increase* all arc capacities along this path in the backward (i.e. sink to source) direction by $C_{\min}$.[2]

Return to Step 1.

[1] The $\leftarrow$ symbol means "takes on the value of." An "=" sign would be inappropriate.

[2] This must be done in case we later wish to reverse the flow on any of these arcs. Before there can be any flow in the opposite direction, we must cancel the current forward flow. The amount of flow which could theoretically exist in the reverse direction *relative to the current forward flow* is the initial reverse direction capacity plus the current forward flow.

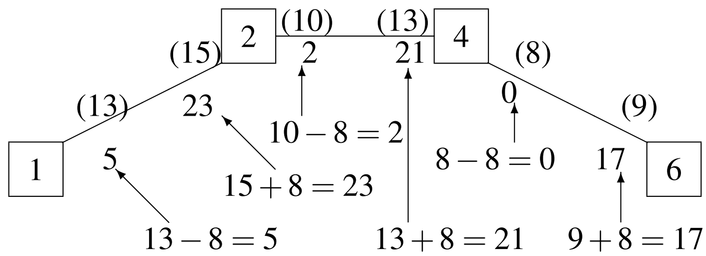


Figure B.2:

B.2 Solution for the Maximal Flow Example

The data comes from the example shown on page 221. In the solution which follows the diagram is re-drawn at each iteration for the sake of clarity. In practice, one diagram would be made with the superseded arc capacities overstriked.

Step 0
Flow = 0

Iteration 1

Step 1
Arbitrarily pick path $\boxed{1} \to \boxed{2} \to \boxed{4} \to \boxed{6}$.
Step 2

$$\begin{aligned} C_{\text{min}} &= \min\{13, 10, 8\} \\ &= 8 \\ \text{Flow} &\leftarrow \text{Flow} + C_{\text{min}} \\ &= 0 + 8 \\ &= 8 \end{aligned}$$

Step 3
The new capacities for the arcs which have been affected are given in Figure B.2.

Iteration 2

Step 1

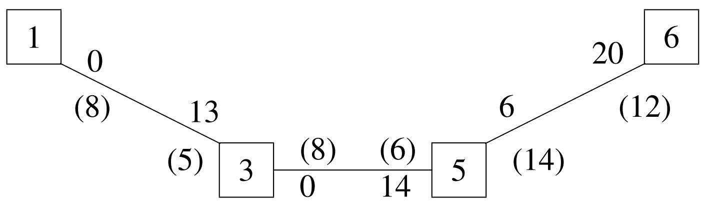


Figure B.3:

Arbitrarily pick path $\boxed{1} \to \boxed{3} \to \boxed{5} \to \boxed{6}$.
Step 2

$$\begin{aligned} C_{\min} &= \min\{8, 8, 14\} \\ &= 8 \\ \text{Flow} &\leftarrow \text{Flow} + C_{\min} \\ &= 8 + 8 \\ &= 16 \end{aligned}$$

Step 3
The new capacities are indicated in Figure B.3

Iteration 3

Step 1
Arbitrarily pick path $\boxed{1} \to \boxed{2} \to \boxed{4} \to \boxed{5} \to \boxed{6}$.
Step 2
Using the capacities which have been updated in the previous iterations, we have:

$$\begin{aligned} C_{\min} &= \min\{5, 2, 8, 6\} \\ &= 2 \\ \text{Flow} &\leftarrow \text{Flow} + C_{\min} \\ &= 16 + 2 \end{aligned}$$

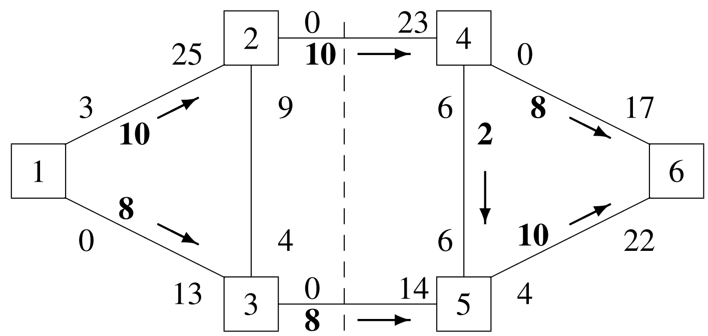


Figure B.4: Flows and Adjusted Arc Capacities

$$= \quad 18$$

Step 3

The new capacities are given in Figure B.4. Each arc flow and its direction is given in boldface on the diagram.

Iteration 4

Step 1

No flow augmenting path can be found, hence the current value of the flow, which is 18, is the maximal flow from $\boxed{1}$ to $\boxed{6}$.

We can see that there is no path with strictly positive capacity, by finding a cut for which the adjusted arc capacities are zero. In this example (see the dashed line on Figure B.4), the capacity of the cut consisting of arcs (2,4) and (3,5) is $0+0=0$. From the max flow/min cut theorem, the maximum amount that can be carried *relative to these arc capacities* is 0. Hence, the current flow of 18 must be the maximal flow from $\boxed{1}$ to $\boxed{6}$.

B.2.1 Second Maximal Flow Example

Now suppose that we wish to know the maximal flow from $\boxed{6}$ to $\boxed{1}$. The original arc capacities are as shown in Figure B.5.

Step 0

Flow = 0

Iteration 1

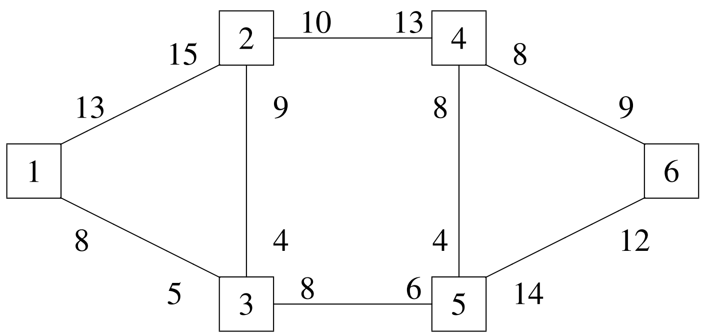


Figure B.5: Second Maximal Flow Example – Arc Capacities

Step 1
Arbitrarily[3] pick path $\boxed{6} \to \boxed{4} \to \boxed{5} \to \boxed{3} \to \boxed{1}$.
Step 2

$$\begin{aligned} C_{\min} &= \min\{9, 8, 6, 5\} \\ &= 5 \\ \text{Flow} &\leftarrow \text{Flow} + C_{\min} \\ &= 0 + 5 \\ &= 5 \end{aligned}$$

Step 3
Subtracting 5 units from the forward direction of this path, and adding 5 units in the reverse direction we obtain Figure B.6.

Iteration 2

Step 1
Arbitrarily pick path $\boxed{6} \to \boxed{4} \to \boxed{2} \to \boxed{1}$.
Step 2

$$C_{\min} = \min\{4, 13, 15\}$$

[3]In solving this problem we will deliberately choose a path at the outset which has an arc for which the flow will later be reversed. This is done so that the procedure for reversing the flow can be illustrated.

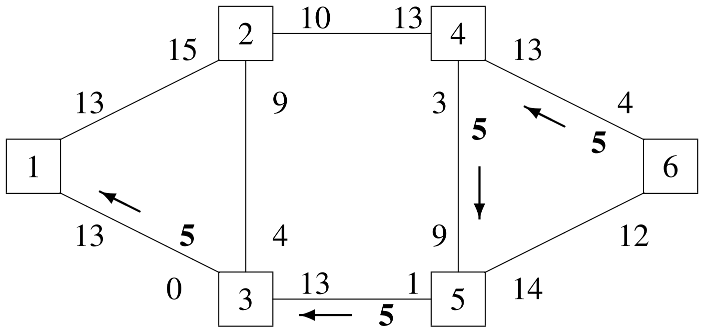


Figure B.6: Five Units of Flow from $\boxed{6}$ to $\boxed{1}$

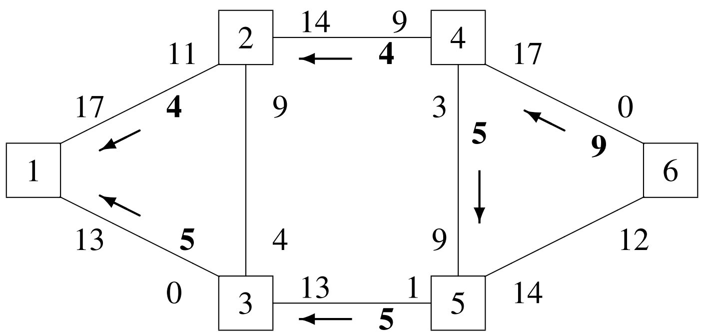


Figure B.7: Nine Units of Flow from $\boxed{6}$ to $\boxed{1}$

$$
\begin{aligned}
&= 4 \\
\text{Flow} &\leftarrow \text{Flow} + C_{\min} \\
&= 5+4 \\
&= 9
\end{aligned}
$$

Step 3

Subtracting 4 units from the forward direction of this path, and adding 4 units in the reverse direction we obtain Figure B.7.

Iteration 3

Step 1

Arbitrarily pick path $\boxed{6} \to \boxed{5} \to \boxed{3} \to \boxed{2} \to \boxed{1}$.

Step 2

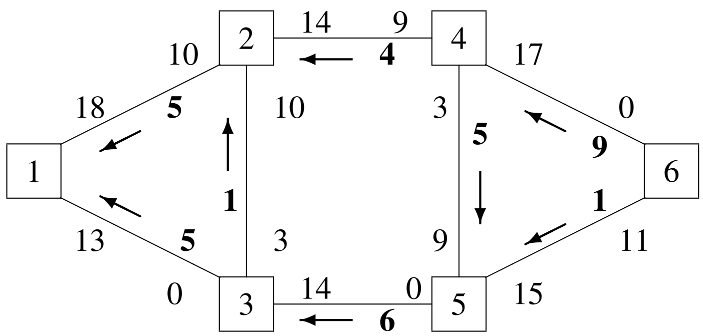


Figure B.8: Ten Units of Flow from $\boxed{6}$ to $\boxed{1}$

$$\begin{aligned} C_{\min} &= \min\{12, 1, 4, 11\} \\ &= 1 \\ \text{Flow} &\leftarrow \text{Flow} + C_{\min} \\ &= 9 + 1 \\ &= 10 \end{aligned}$$

Step 3
Subtracting 1 unit from the forward direction of this path, and adding 1 unit in the reverse direction we obtain Figure B.8.

Iteration 4

Step 1
Arbitrarily pick path $\boxed{6} \to \boxed{5} \to \boxed{4} \to \boxed{2} \to \boxed{1}$.
Step 2

$$\begin{aligned} C_{\min} &= \min\{11, 9, 9, 10\} \\ &= 9 \\ \text{Flow} &\leftarrow \text{Flow} + C_{\min} \\ &= 10 + 9 \\ &= 19 \end{aligned}$$

Step 3

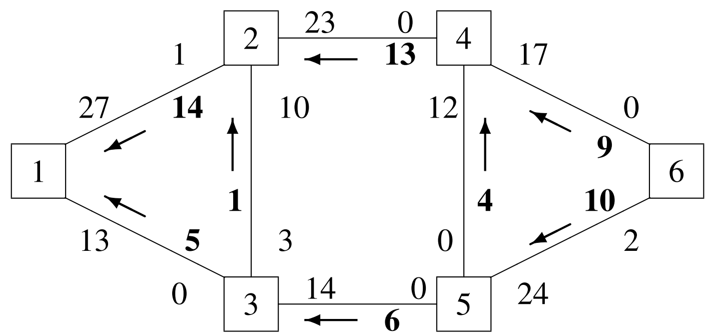


Figure B.9: Nineteen Units of Flow from $\boxed{6}$ to $\boxed{1}$

Subtracting 9 units from the forward direction of this path, and adding 9 units in the reverse direction we obtain Figure B.9.

Iteration 5

Step 1

No remaining paths with forward direction capacity.

Therefore, maximal flow = 19 units.

Note that at iteration 4 the flow was reversed, with 9 units being sent from $\boxed{6}$ to $\boxed{1}$ via $\boxed{5}$, $\boxed{4}$, and $\boxed{2}$. Had we not increased the reverse capacity from 4 units to 9 units at iteration 1, the reversion at iteration 4 would not have been possible.

B.2.2 Limiting Cuts

There are two cuts which limit the flow to 19 units in this example. They are:

	Cut Arcs	Cut Capacity
(a)	(4,2), (5,3)	13 + 6 = 19
(b)	(5,3), (5,4), (6,4)	6 + 4 + 9 = 19

B.3 Shortest Path Algorithm

As an alternative to solving a shortest path problem by formulating and then solving it as a linear optimization problem, we can create a dedicated procedure for this problem called the "Shortest Path[4] Algorithm." This algorithm is based on

[4] Also called the "shortest *route* algorithm."

a labelling procedure. Each node is labelled in the form (x, y), where x is the distance of the shortest path *found so far* from the starting point to the node of interest, and y is the number of the *predecessor* node (i.e. the immediately previous node) on that path. The label is *permanent* if x is known to be the shortest distance, otherwise the label is *temporary* and may subsequently be *updated*.

Step 1: At the outset, the starting node is permanent; the rest are non-permanent.

Step 2: For the most recently permanently labelled node, which we call node y, determine all of the non-permanent nodes which can be reached directly from node y.

Step 3: Assign a temporary label in the form (x, y) to each of these reachable nodes using the permanently labelled node as the predecessor (y), and where x is the distance from the starting node to node y plus the distance from node y to the reachable node.

Step 4: For each of the temporarily labelled nodes, keep only the temporary label which has the minimum value of x.

Step 5: Amongst *all* of the temporarily labelled nodes find the smallest distance x in the corresponding label and designate this node and this label as permanent.[5]

Step 6: If the destination node has been permanently labelled, then STOP. Otherwise, return to Step 2.

B.3.1 Solution for the Shortest Path Example

Applying this algorithm to the network described on page 226, the off-the-diagram variation proceeds as follows:

Iteration 1:
Designate the starting node as permanently labelled.

[5] When there is a tie among two or more nodes for the smallest x, then both (or all) tied nodes would become permanently labelled. Steps 2 and 3 would then involve more than one y-type node.

Permanently Labelled Node(s)	Temporarily Labelled Node(s)	Temporary Label
1	2	(40,1)
1	3	(58,1)
1	4	(30,1)*

The symbol * is used to designate the label which became permanent. At this point we know that the shortest distance to node 4 is 30. In addition, to arrive at any other temporarily labelled node we must travel a distance greater than 30.

Iteration 2:

Node 4 has a permanent label so we need to assign temporary labels to all the nodes which can be reached from node 4.

Permanently Labelled Node(s)	Temporarily Labelled Node(s)	Temporary Label	Minimum Label
1	2	(40,1)	(40,1)*
1	3	(58,1)	—
4	3	(46,4)	(46,4)
4	6	(50,4)	(50,4)

Note that we always use the cumulative distance. Thus for going from 4 to 3 we use 30 + 16 = 46 and from 4 to 6 we use 30 + 20 = 50.

Iteration 3:

Node 2 has a permanent label so we need to assign temporary labels to all the nodes which can be reached from node 2.

Permanently Labelled Node(s)	Temporarily Labelled Node(s)	Temporary Label	Minimum Label
4	3	(46,4)	(46,4)*
4	6	(50,4)	(50,4)
2	3	(52,2)	—
2	5	(110,2)	(110,2)

Iteration 4:

Node 3 has a permanent label so we need to assign temporary labels to all the nodes which can be reached from node 3.

Permanently Labelled Node(s)	Temporarily Labelled Node(s)	Temporary Label	Minimum Label
4	6	(50,4)	(50,4)*
2	5	(110,2)	—
3	5	(101,3)	(101,3)
3	6	(71,3)	—-
3	7	(111,3)	(111,3)

Iteration 5:

Node 6 has a permanent label so we need to assign temporary labels to all the nodes which can be reached from node 6.

Permanently Labelled Node(s)	Temporarily Labelled Node(s)	Temporary Label	Minimum Label
3	5	(101,3)	(101,3)
3	7	(111,3)	—
6	7	(85,6)	(85,6)*

Since $\boxed{7}$ is permanently labelled at this iteration, we have the solution to the original problem. The shortest distance from $\boxed{1}$ to $\boxed{7}$ is a distance of 85 units, and the path can be "traced back" using the y's.

The shortest path from $\boxed{1}$ to $\boxed{7}$ lies through $\boxed{6}$.

The shortest path from $\boxed{1}$ to $\boxed{6}$ lies through $\boxed{4}$.

Therefore the shortest path is $\boxed{1} \to \boxed{4} \to \boxed{6} \to \boxed{7}$ with a distance of 85 units. If we wish to find the shortest path from $\boxed{1}$ to *all* other nodes, we need only to continue the iterations until all nodes have become permanently labelled.

Iteration 6:

Node 7 has a permanent label so we need to assign temporary labels to all the nodes which can be reached from node 7.

Permanently Labelled Node(s)	Temporarily Labelled Node(s)	Temporary Label	Minimum Label
3	5	(101,3)	—
7	5	(100,7)	(100,7)*

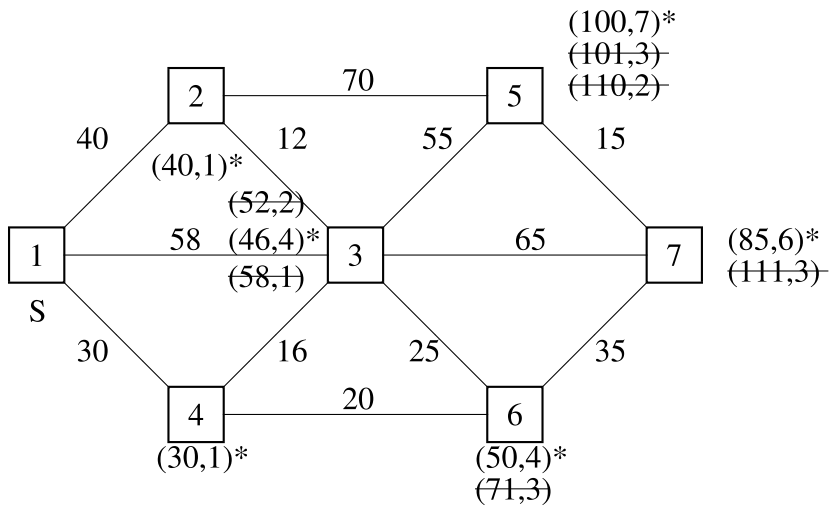


Figure B.10: Solution Performed on the Network

Thus the shortest paths from $\boxed{1}$ to all other nodes are:

From $\boxed{1}$ to	Distance	Path
$\boxed{2}$	40	$1 \to 2$
$\boxed{3}$	46	$1 \to 4 \to 3$
$\boxed{4}$	30	$1 \to 4$
$\boxed{5}$	100	$1 \to 4 \to 6 \to 7 \to 5$
$\boxed{6}$	50	$1 \to 4 \to 6$
$\boxed{7}$	85	$1 \to 4 \to 6 \to 7$

This solution can be determined right on the network. In this visual variation of the shortest path algorithm, the labels are crossed out as they become superceded [6] (i.e. as x and y change). The labels are marked with an asterisk when the status changes from temporary to permanent. The starting node is labelled with an "S".

[6]We have indicated this by drawing a horizontal line through the label.

Appendix C

Integer Extensions

Here we present two more advanced integer models, and the branch-and-bound method for solving integer models.

C.1 Advanced Models

C.1.1 A Capacity Planning Problem

Description

An electrical utility has five projects available. Some or all of these projects will be built over a ten year planning horizon. The utility can choose which projects it wishes to undertake, and the year in which each undertaken project commences to produce electricity. However, because of construction time, the earliest that project i can commence producing electricity is in year p_i, where $1 \leq p_i \leq 10$ (for example, if $p_3 = 6$, then the utility can choose project 3 to commence in year 6 or in any subsequent year). If undertaken, the ith project will incur a capital cost of c_i dollars in the year in which the project commences, and will contribute a capacity of e_i units of electrical energy per annum. (There is no cost beyond the commencement year of each project, but it continues to produce electricity at the same rate in each subsequent year). The utility has a capital budget of b_t dollars in year t, where $1 \leq t \leq 10$. Any unspent capital funds are accumulated from year to year. (Hence, for example, in year 5 the company may spend any leftover money at the end of year 4 plus b_5.) There is a current year (year 0) surplus of 40 units of energy (i.e. the current capacity is 40 units greater than the demand). The increase in demand from year $t-1$ to year t will be d_t units of energy. In each year, the capacity cannot be less than the demand.

Ignoring the time value of money, we wish to formulate a model which will minimize the surplus of generating capacity in the tenth year.

Note: the p's, c's, e's, b's and d's are parameters, not variables.

Formulation

The problem has two entities which resemble inventories, at least as far as the modelling is concerned. First, there are the unspent capital funds at the end of each year. Since these are unknown, we define:

$$U_t = \text{ amount of unspent funds at the end of year } t, \quad t = 0, \ldots, 10$$

Secondly, there is the surplus of generating capacity at the end of each year. Because the surpluses are unknown, we represent them using variables. These variables will have a non-negativity restriction, which is precisely what we want since the "capacity cannot be less than the demand." Hence we define:

$$S_t = \text{ amount of surplus capacity at the end of year } t, \quad t = 0, \ldots, 10$$

We must decide if and when each project *commences*, for which we define:

$$Y_{it} = \left\{ \begin{array}{ll} 1 & \text{if project } i \text{ commences in year } t \\ 0 & \text{otherwise} \end{array} \right\} \begin{array}{l} i = 1,2,3,4,5 \\ t = p_i, \ldots, 10 \end{array}$$

Note that Y_{it} is only defined for those situations were it *could* take on the value of 1. (If $t < p_i$, then the project is definitely not built in that year.)

The objective, which is to minimize the ending surplus of generating capacity, is simply

$$\text{minimize} \quad S_{10}$$

The initial level of capital funds is $U_0 = 0$. In year t, the amount of capital which can be invested is the initial level of capital funds, which is U_{t-1}, plus that year's capital allocation, which is b_t. The capital cost of project i in year t is $c_i Y_{it}$. The total capital spent in year t is found by summing over all projects. This sum is

$$\sum_{\substack{\text{all } i \text{ such} \\ \text{that } p_i \leq t}} c_i Y_{it}$$

To avoid clutter, we will shorten this to

$$\sum_i c_i Y_{it}$$

The amount of leftover capital at the end of year t is U_t. Balancing these four amounts we obtain

$$U_{t-1} + b_t = \sum_i c_i Y_{it} + U_t$$

Re-arranging so that the variables are on the left and the parameter is on the right we obtain, for $t = 1, \ldots, 10$,

$$-U_{t-1} + \sum_i c_i Y_{it} + U_t = b_t$$

The initial surplus of generating capacity is given as $S_0 = 40$. In each year t, the surplus at the outset of the year, plus the newly installed capacity during that year, must equal the increase in demand during that year plus the ending surplus. The surplus at the outset of year t is S_{t-1}. The capacity installed during year t is

$$\sum_{\substack{\text{all } i \text{ such} \\ \text{that } p_i \le t}} e_i Y_{it}$$

Again, to avoid clutter, we will shorten this to

$$\sum_i e_i Y_{it}$$

The increase in demand during the year is d_t, and the capacity surplus at the end of year t is S_t. Hence

$$S_{t-1} + \sum_i e_i Y_{it} - S_t = d_t \quad t = 1, \ldots, 10$$

Note that the *absolute* level of demand does not need to be tracked.

Each project can only be built once, hence

$$\sum_{t=p_i}^{10} Y_{it} = 1 \quad i = 1, \ldots, 5$$

Finally, we require that each U_t and each S_t be non-negative ($t = 0, \ldots, 10$), and each $Y_{it} \in \{0, 1\}$ for $i = 1, 2, 3, 4, 5$ and $t = p_i, \ldots, 10$. Putting all this together we obtain:

$$
\begin{array}{rrcll}
\text{minimize} & S_{10} & & & \\
\text{subject to} & & & & \\
(1) & U_0 & = & 0 & \\
(2),\ldots,(11) & -U_{t-1} + \sum_i c_i Y_{it} + U_t & = & b_t & (t = 1,\ldots,10) \\
(12) & S_0 & = & 40 & \\
(13),\ldots,(22) & S_{t-1} + \sum_i e_i Y_{it} - S_t & = & d_t & (t = 1,\ldots,10) \\
(23),\ldots,(27) & \sum_{t=p_i}^{10} Y_{it} & \leq & 1 & (i = 1,\ldots,5) \\
 & U_t, S_t & \geq & 0 & (t = 0,\ldots,10) \\
 & Y_{it} \in \{0,1\} & & & \left\{ \begin{array}{l} i = 1,\ldots,5 \\ t = p_i,\ldots,10 \end{array} \right\}
\end{array}
$$

C.1.2 A Journey by Rail

Description

A railway buff wishes to travel eastbound from the Atlantic Ocean to the Pacific Ocean. He will begin at Lisbon, Portugal (station 1) and finish at Vladivostok, Russia (station 20) via a pre-determined route. Over a 30 day month, a schedule has been published by the railway companies indicating which stations have daytime connecting services on which days. Parameter s_{ijt} is 1 if there is a service which goes from station i to station j ($j > i$) on day t, and is 0 otherwise. He will only travel on one service per day, will always travel eastbound, and will never travel at night. He wishes to complete the journey within 30 days. He need not travel every day.

Formulation

We see that from reading the problem there is a binary choice associated with each service. Since parameter s has triple subscription, we will let the associated decision variable have triple subscription. We define[1]

$$Y_{ijt} = \left\{ \begin{array}{ll} 1 & \text{if he travels from station } i \\ & \text{to station } j \text{ on day } t \\ 0 & \text{otherwise} \end{array} \right\} \qquad \begin{array}{l} i = 1, \ldots, 19 \\ j = i+1, \ldots, 20 \\ t = 1, \ldots, 30 \end{array}$$

The objective is to minimize the day of arrival at Vladivostok ($j = 20$). What we want is the value of t for which $Y_{i20t} = 1$. (The context requires that each other Y_{i20t} be 0.) Since each Y_{i20t} is either 0 or 1, the product tY_{i20t} is either 0 or t. Hence

$$\text{Day of Arrival at Vladivostok} = \sum_{i=1}^{19} \sum_{t=1}^{30} tY_{i20t}$$

which is precisely what we wish to minimize. The objective function is therefore:

$$\text{minimize} \quad \sum_{i=1}^{19} \sum_{t=1}^{30} tY_{i20t}$$

We now examine the constraints. Since he cannot travel on a service if it does not exist, an obvious set of constraints is that Y_{ijt} must be 0 if s_{ijt} is 0. (If $s_{ijt} = 1$, then Y_{ijt} can be either 0 or 1.) Hence

$$Y_{ijt} \leq s_{ijt} \qquad i = 1, \ldots, 19 \quad j = i+1, \ldots, 20 \quad t = 1, \ldots, 30$$

While this is correct, we have created $(19+18+\cdots+2+1)30 = 5700$ constraints! For now, we will continue to formulate the model as we had been doing, but we will later examine an alternate formulation.

At Lisbon (station 1), he must leave to go somewhere on one of the days of the month. Hence we have the equality constraint

$$\sum_{j=2}^{20} \sum_{t=1}^{30} Y_{1jt} = 1$$

At Vladivostok (station 20), he must arrive from somewhere on one of the days of the month. Hence we have the equality constraint

$$\sum_{i=1}^{19} \sum_{t=1}^{30} Y_{i20t} = 1$$

[1]Note the index for j which defines it only for $j > i$.

We now consider the intermediate stations (stations 2 to 19 inclusive). If he ends a journey at a particular station, then he must begin a journey from this station on a subsequent day. Otherwise, he is on a train which does not stop at this station, or if it does, he does not alight from it. Consider a station k where $2 \leq k \leq 19$, and a day d, where $2 \leq d \leq 30$. If he arrives at k from i *before* day d, then he must leave k for j on or after day d. He arrives at k from i before day d if and only if

$$\sum_{i=1}^{k-1} \sum_{t=1}^{d-1} Y_{ikt} = 1$$

He leaves k for j on or after day d if and only if

$$\sum_{j=k+1}^{20} \sum_{t=d}^{30} Y_{kjt} = 1$$

Subtraction gives us both the case of ending a journey at k and the case of *not* ending a journey at k at once.

$$\sum_{i=1}^{k-1} \sum_{t=1}^{d-1} Y_{ikt} - \sum_{j=k+1}^{20} \sum_{t=d}^{30} Y_{kjt} = 0 \qquad k = 2, \ldots, 19 \quad d = 2, \ldots, 20$$

If he ends a journey at k, then we have $1 - 1 = 0$, and if he does not end a journey at k, then we have $0 - 0 = 0$. There is no concern about the constraint being met by say $2 - 2 = 0$, because this is prevented, in conjunction with the preceding set of equations, by the beginning equation at Lisbon, and the ending equation at Vladivostok.

Finally, each $Y_{ijt} \in \{0,1\}$. The complete formulation is therefore:

$$
\begin{array}{rrcll}
\text{minimize} & \displaystyle\sum_{i=1}^{19}\sum_{t=1}^{30} tY_{i20t} & & & \\
\text{subject to} & & & & \\
(1) & \displaystyle\sum_{j=2}^{20}\sum_{t=1}^{30} Y_{1jt} & = & 1 & \\
(2) & \displaystyle\sum_{i=1}^{19}\sum_{t=1}^{30} Y_{i20t} & = & 1 & \\
(3),\ldots,(344) & \displaystyle\sum_{i=1}^{k-1}\sum_{t=1}^{d-1} Y_{ikt} - \sum_{j=k+1}^{20}\sum_{t=d}^{30} Y_{kjt} & = & 0 & \left\{\begin{array}{l} k=2,\ldots,19 \\ d=2,\ldots,20 \end{array}\right\} \\
(345),\ldots,(6044) & Y_{ijt} & \leq & s_{ijt} & \left\{\begin{array}{l} i=1,\ldots,19 \\ j=i+1,\ldots,20 \\ t=1,\ldots,30 \end{array}\right\} \\
 & Y_{ijt} \in \{0,1\} & & & \left\{\begin{array}{l} i=1,\ldots,19 \\ j=i+1,\ldots,20 \\ t=1,\ldots,30 \end{array}\right\}
\end{array}
$$

An Alternate Formulation

Clearly, the number of constraints given above is highly excessive. The way to avoid this is simply to never mention a particular Y_{ijt} if the corresponding s_{ijt} is 0. Not only does this remove 5700 constraints, the objective function and the other constraints are shortened, because the non-relevant Y_{ijt}'s are no longer mentioned. For example, while there are $19 \times 30 = 570$ potential services which end at Vladivostok, perhaps only 40 exist. Hence, the objective function needs only these 40 terms instead of all 570. Since we are using symbolic notation, we need to indicate this fact. If we think of Y as a set, and S as the set of all $s_{ijt} = 1$, then $Y \in S$.

We will put a restriction indicating this fact under the word *minimize*.

$$
\begin{array}{rrcl}
\underset{Y \in S}{\text{minimize}} & \displaystyle\sum_{i=1}^{19}\sum_{t=1}^{30} tY_{i20t} & & \\
\text{subject to} & & & \\
(1) & \displaystyle\sum_{j=2}^{20}\sum_{t=1}^{30} Y_{1jt} & = & 1 \\
(2) & \displaystyle\sum_{i=1}^{19}\sum_{t=1}^{30} Y_{i20t} & = & 1 \\
(3),\ldots,\ (344) & \displaystyle\sum_{i=1}^{k-1}\sum_{t=1}^{d-1} Y_{ikt} - \sum_{j=k+1}^{20}\sum_{t=d}^{30} Y_{kjt} & = & 0 \quad \left\{ \begin{array}{l} k = 2,\ldots,19 \\ d = 2,\ldots,20 \end{array} \right\} \\
& Y_{ijt} \in \{0,1\} & & Y \in S
\end{array}
$$

C.2 A Branch and Bound Algorithm

Early work on algorithms for integer optimization began with the *cutting plane* method (1958), and the *branch and bound* method (1960). Much research continued, especially on the branch and bound algorithm.[2] More recently, these methods have been combined to give, a least for certain examples, the ability to solve large-scale problems.[3] The level and scope of the text restricts our discussion to the general principles of the branch-and-bound algorithm.

C.2.1 Descriptive Overview

The algorithm will be described formally later. This statement will suffice as to the *how* of the algorithm. It is the purpose, however, of the section to give an idea of the *why* of the methodology.

For a *pure* integer model, at least, it may seem that a complete enumeration of all possible solutions would be a possible approach for solving the model optimally. We can think of this as a decision tree, where the first decision is to choose

[2]For references on this early research see Geoffrion, A.M. and R.E. Marsten, "Integer Programming Algorithms: A Framework and State-of-the-Art Survey," *Management Science*, **18** (1972), 465-491.

[3]Van Roy, T.J. and L.A. Wolsey, "Solving Mixed 0-1 Programs by Automatic Reformulation," *Operations Research*, **35** (1987), 45-57.

variable 1 to be 0 or 1, the second decision is to choose variable 2 to be 0 or 1, and so on. The ending nodes of this tree correspond with a particular solution. For example, if a model has four 0/1 variables, then there are $2^4 = 16$ possible solutions. We could examine each of these for feasibility, and then choose the best solution from amongst the feasible solutions. While this works in theory, in practice the time to examine the solutions becomes prohibitively large as the number of variables increases, because this growth is exponential. Even if we were to ignore the fact that the time to examine each solution increases as the size of the problem increases (which of course only makes things worse), the effect of problem size on computer time is staggering. Where n is the number of 0/1 variables, if a computer examines 1,000,000 solutions per second we obtain:

n	2^n	Time	
10	1,024	1.024	millisecond
20	1,048,576	1.048	second
30	1,073,741,824	17.90	minutes
40	1.0995×10^{12}	12.73	days
50	1.1259×10^{15}	35.68	years

Hence, complete enumeration is impractical except for small examples. This motivates us to seek a better algorithm. Unfortunately, there is no algorithm which, for an integer model of arbitrary structure, can in the worst case scenario break away from the exponential nature which plagues complete enumeration. However, an algorithm may give the optimal solution in a reasonable time for most medium sized problems.[4]

Continuing the metaphor of a tree, the branch and bound algorithm seeks to reduce the number of solutions examined by judiciously pruning many of the tree's potential branches. The tree is built by the algorithm adding branches when necessary. When the algorithm identifies that adding branches is not necessary, we can think of these branches and all the branches coming off these branches as having been pruned. In the worst case, nothing is pruned, and hence the number of twigs is the same as it is for complete enumeration, but usually we can do much better than this. Unlike a deterministic decision tree, which connects decisions, the branch and bound tree connects solutions. This algorithm contains a second metaphor, that of parents and children. Each person is a linear model with a corresponding solution, with each child being connected to its parent by a

[4]Some problems are simply not solvable in our lifetimes. Chess is an example – the tree size becomes enormous in part because there are often many legal moves at each player's turn.

branch. Solving the original model is called the *master* problem; the obtaining of a solution to a linear model is called a *sub-problem*.

Since our integer models only differ from the standard linear models in that some of the variables have integrality restrictions, the starting point of the algorithm is to solve the model as if these restrictions were replaced by the standard non-negativity restrictions. We call this the *relaxed* model since we have relaxed the integrality restrictions. The relaxed model is of course solvable by the simplex algorithm, which we already know how to do. If the optimal solution to the relaxed model obeys the integrality restrictions anyway, then we have the optimal solution to the integer model.[5] Otherwise, we must continue.

We can think of the original relaxed model as sub-problem 1, with an OFV which is labelled OFV(1). If the master problem is a maximization model, then OFV(1) represents an *upper bound* (UB) to OFV*, since when the integrality restrictions are added it can only impair[6] the OFV. Conversely, if the master problem is a minimization model, then OFV(1) represents a *lower bound* (LB) to OFV*.

Sometimes, a *feasible* solution to the integer model is trivially obvious. In such a situation, we have identified a solution with which all other solutions can be compared. We will always compare a new solution with the best one found so far, called the *incumbent* (I) solution. At the outset, the first feasible solution found serves as the incumbent. For a maximization problem, OFV$^* \geq$ OFV(I). For a minimization problem, OFV$^* \leq$ OFV(I). If there is no obvious feasible solution, we can always state that OFV$^* > -\infty$ for a maximization problem, and OFV$^* < \infty$ for a minimization problem. The incumbent establishes a *lower bound* for the value of OFV* for a max model; it establishes an *upper bound* for the value of OFV* for a min model.

Putting all this together we see that

$$\begin{array}{rccccccc}
\text{max model} & -\infty & < & \text{OFV(I)} & \leq & \text{OFV}^* & \leq & \text{OFV(1)} \\
\text{min model} & \text{OFV(1)} & \leq & \text{OFV}^* & \leq & \text{OFV(I)} & < & \infty
\end{array}$$

Whether max or min, the idea is that OFV* is bounded:

$$\text{LB} \leq \text{OFV}^* \leq \text{UB}$$

[5]If the linear solution provided by the computer is not linear, but if multiple optima exist, then one of these alternate solutions may be integer.

[6]If multiple optima exist it may stay the same, hence the bound is not a strict inequality.

As the algorithm progresses, higher values for LB and/or lower values for UB will be discovered until we reach a point where LB = OFV* = UB. The solution for which this equation is true is therefore the optimal solution to the master problem.

The two children are like their parent except that each one has either an extra constraint or a modified constraint. The children are created so that there is no overlap, but at the same time no valid solution is excluded. For example, suppose that the original model requires that $Y_5 \in \{0, 1, 2, \ldots\}$. If in the current solution $Y_5 = 3.481$, we can break this apart by adding the constraint $Y_5 \leq 3$ to one child, and by adding the constraint $Y_5 \geq 4$ to the other. The optimal solution must then obey the constraint set of one of the two children.

The operation which creates the children is called *branching*. This process of branching can be used to create grand-children and so on. If it can be shown that the *best* solution amongst a sub-problem and all its descendants (given by this sub-problem's OFV) is *worse* than an already known feasible solution to the original integer problem (given by the OFV of the incumbent), then this sub-problem and all its descendants can be eliminated from consideration. The determination of this elimination is called *bounding*; the elimination itself is often called *fathoming*.

It is partly the fathoming which allows the branch and bound algorithm to save time compared with complete enumeration, but finding integer or infeasible sub-problems helps as well. There is no point in examining any of the descendants of a sub-problem if any of these three conditions holds:

1. the solution to the sub-problem is infeasible

2. the OFV of the sub-problem is worse than the OFV of the incumbent ('worse' means 'greater than' for a min problem, and 'less than' for a max problem).

3. the solution to the sub-problem is valid for the original model (i.e. if all variables which must be 0/1 or general integer are precisely that)

If the third condition holds, the solution is compared with the incumbent. If the new one is better (lower OFV for a min model, higher OFV for a max model), then the current sub-problem becomes the new incumbent.

As sub-problems are solved, unless one of the three stated conditions holds, two children sub-problems are created. This creates or adds to a queue of sub-problems waiting to be solved. When the queue becomes empty, the best integer solution found to that point, the incumbent, is declared to be the optimal solution to the original model. (Equivalently, the upper and lower bounds have converged.)

This optimal solution may have been found early in the search procedure, however, it is only after the queue of sub- problems vanishes that we can say for certain that it is optimal.

We now give two examples to illustrate the algorithm. After that follows a formal statement of the branch and bound algorithm for integer optimization.

C.2.2 A Maximization Example

$$
\begin{array}{rrcrcrcrcr}
\text{maximize} & 3.1X & + & 2.7Y & + & 8.4G_1 & + & 5.9G_2 & & \\
\text{subject to} & & & & & & & & & \\
(1) & 1.7X & + & 8Y & + & G_1 & + & 6G_2 & \leq & 20 \\
(2) & 2X & + & 5Y & + & 4G_1 & + & 3G_2 & \leq & 31 \\
(3) & 0.8X & + & 0.4Y & + & 2G_1 & + & 1.3G_2 & \leq & 15 \\
(4) & & & & & G_1 & + & 2G_2 & \leq & 10 \\
(5) & 2X & + & 7Y & & & & & \leq & 12 \\
(6) & & & & & 2G_1 & + & 7G_2 & \leq & 25
\end{array}
$$

$$X \geq 0,\ \ Y \in \{0,1\},\ \ G_1, G_2 \in \{0,1,2,\ldots\}$$

This example contains a continuous variable, a 0/1 variable, and two general integer variables.

We now relax the integrality restrictions and replace them with the standard non-negativity restrictions $Y \geq 0$, $G_1 \geq 0$, and $G_2 \geq 0$. We call this new model *sub-problem 1*. Solving this as an ordinary linear optimization problem we obtain:

$$
\begin{array}{rcl}
\text{OFV}(1) & = & 63.846060 \\
X & = & 0.000000 \\
Y & = & 0.070539 \\
G_1 & = & 6.342324 \\
G_2 & = & 1.759336
\end{array}
$$

Since (1) is not a valid solution for the original model we must continue. Since each constraint is of the less-than-or-equal-to type with only positive structural coefficients, we can round each variable downwards to obtain the following feasible solution to the original model:

$$\text{OFV}(1) \;=\; 56.3$$

$$\begin{aligned} X &= 0.0 \\ Y &= 0 \\ G_1 &= 6 \\ G_2 &= 1 \end{aligned}$$

(Note that the value of the continuous variable X is written as a real number.) Hence the incumbent solution has an OFV of 56.3, and

$$\text{LB} = 56.3 \leq \text{OFV}^* \leq 63.84606 = \text{UB}$$

We need to create two descendant sub-problems numbered (2) and (3). To accomplish this we could choose any variable which is supposed to be integer but which is not, and in one sub-problem restrict this variable to be no more than the integer number just below the current value of this variable, and in the other sub-problem restrict this variable to be no less than the integer number just above the current value of this variable.

For example, in sub-problem (1) $Y = 0.070539$. This variable is supposed to be either 0 or 1. Hence we could require Y to be 0 in subproblem (2), and require it to be 1 in sub-problem (3). This would be done by adding an equality constraint to sub-problem (1). Another choice is variable G_1, which is currently 6.342324. If we were to choose this variable then we would add the constraint $G_1 \leq 6$ to sub-problem (2), and add the constraint $G_1 \geq 7$ to sub-problem (3). Finally, we could choose to branch on G_2, which is currently 1.759336, by adding $G_2 \leq 1$ to sub-problem (2), and $G_2 \geq 2$ to sub-problem (3). [We would never branch on X, because it is a continuous variable.]

There is no exact rule for deciding which variable should be chosen. One strategy is to choose the integer variable whose current value is furthest away from the nearest integer number. [Or equivalently, choose the integer variable whose fractional component is closest to 0.5] For the example at hand we have 0.07 for Y, 0.34 for G_1, and $1 - 0.76 = 0.24$ for G_2. Hence, using this strategy we would choose to branch on G_1.

Sub-problem (2) is the same as sub-problem (1) except that we add the constraint $G_1 \leq 6$; sub-problem (3) is the same as sub-problem (1) except that we add the constraint $G_1 \geq 7$. These sub-problems are put into a queue of sub-problems waiting to be solved. The information which needs to be stored concerning the sub-problems in the queue is the sub-problem number, the number of its parent, the new constraint which differentiates it from its parent, and the OFV of the parent. Hence the queue is:

Sub-problem Number	Parent	New Constraint	OFV (Parent)
2	1	$G_1 \leq 6$	63.846
3	1	$G_1 \geq 7$	63.846

While there are many rules that one could use concerning the selection of a sub-problem from the queue, a reasonable one for our purposes is to select the sub-problem whose parent has the most favourable OFV (highest in a max problem, lowest in a min problem), and break a tie by choosing the lower numbered sub-problem (FIFO). Obviously, (2) and (3) share the same parent so we begin with (2).

Solving (2) yields the solution:

$$\begin{aligned}
\text{OFV}(2) &= 63.571430 \\
X &= 0.714286 \\
Y &= 0 \\
G_1 &= 6 \\
G_2 &= 1.857143
\end{aligned}$$

This solution does not fall into any of the three categories which would lead to abandoning a further search along this path, i.e. the solution is not infeasible, it is not worse than the incumbent ($63.57143 \nleq 56.3$), and it is not a feasible solution for the original problem. Therefore, we create two descendants, branching on variable G_2. Sub-problem (4) is the same as *its* parent (which is (2)), except that we add the constraint $G_2 \leq 1$; sub-problem (5) is like (2) except that we add the constraint $G_2 \geq 2$.

To keep all this straight, we will draw a tree with boxes to represent each sub-problem. Each box contains the sub-problem number (top line, left), the iteration number (top line, right), the value of the LB for a max model (UB for a min model) just prior to solving the sub-problem (second line), the OFV and the values of the variables (third to seventh lines), and finally a statement (eighth line) concerning the variable upon which to branch, or a reason for not branching. In the latter case, we will use the words "infeasible" if the relaxation is infeasible, "OFV $>$ UB" (minimization) or "OFV $<$ LB" (maximization), or "valid" to mean that all constraints and integrality restrictions of the original model are satisfied.

At the moment, the boxes for (1) and (2) are complete, and the boxes for (3), (4), and (5) are drawn but empty. A line connects each box (except (1)) to its

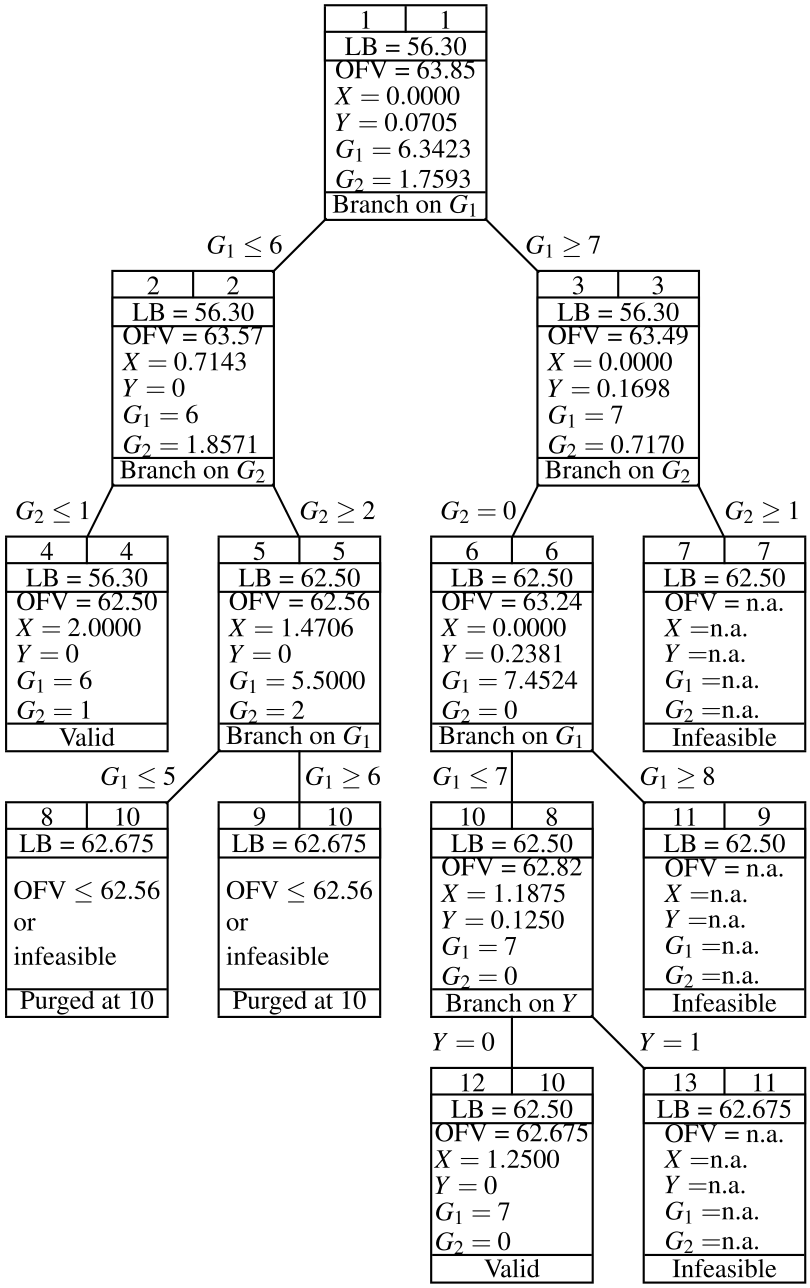


Figure C.1: Branch and Bound Tree for a Maximization Example parent, next to which is a statement of the constraint which has been added to the parent. The completed tree is shown in Figure C.1.

Now sub-problems 3, 4, and 5 are in the queue. Using the stated rule (choose the parent with the highest OFV for a max model), we choose sub-problem 3 (63.846 vs 63.571). This is the same as sub-problem 1, except that we add the constraint $G_1 \geq 7$.[7]

Solving sub-problem 3 we obtain:

$$\begin{aligned} \text{OFV}(3) &= 63.48868 \\ X &= 0.0 \\ Y &= 0.169811 \\ G_1 &= 7 \\ G_2 &= 0.716981 \end{aligned}$$

This solution is feasible, is not worse than the incumbent (since $63.49 > 56.3$), and is not integer, and hence we branch. Since G_2 is $1 - .717 = 0.283 > .170$ we branch on it rather than Y. We create two descendants: sub-problem 6 adds the constraint $G_2 = 0$ to sub-problem 3, while sub-problem 7 adds the constraint $G_2 \geq 1$ to sub-problem 3. There are now four sub-problems in the queue:

Sub-problem Number	Parent	New Constraint	OFV (Parent)
4	2	$G_2 \leq 1$	63.571
5	2	$G_2 \geq 2$	63.571
6	3	$G_2 = 0$	63.489
7	3	$G_2 \geq 1$	63.489

For a max model, the OFV(parent) in the queue represents an *upper bound* to OFV* (for a min model it represents a lower bound). The new upper bound is therefore 63.571, and

$$56.3 \leq \text{OFV}^* \leq 63.571$$

As the iterations proceed, the lower and upper bounds will converge to the OFV of the optimal integer solution.

[7] As a practical measure, since it consumes a lot of memory to store every model, and since sub-problem 2 was the last one solved, sub-problem 3 can be obtained by altering the last constraint of sub-problem 2 from $G_1 \leq 6$ to $G_1 \geq 7$. Doing this can be tricky when the last sub-problem solved is nowhere near the current sub-problem on the tree.

Examining the queue we remove sub-problem 4. Solving we obtain:

$$\begin{aligned} \text{OFV}(4) &= 62.5 \\ X &= 2.0 \\ Y &= 0 \\ G_1 &= 6 \\ G_2 &= 1 \end{aligned}$$

Since $Y \in \{0,1\}$, and $G_1, G_2 \in \{0,1,2,\ldots\}$ as the original model requires, we do not branch further down this path. Moreover, this feasible solution to the original problem is better than the incumbent (62.5 $>$ 56.3), hence sub-problem 4 becomes the new incumbent. We therefore have a new lower bound of 62.5. Either this is the optimal solution, or the optimal solution which remains to be discovered is better than this. Hence:

$$62.5 \leq \text{OFV}^* \leq 63.571$$

Hence the optimal solution is at most 1.071 units higher than the incumbent. We now solve sub-problem 5, obtaining:

$$\begin{aligned} \text{OFV}(5) &= 62.559 \\ X &= 1.47058 \\ Y &= 0 \\ G_1 &= 5.50000 \\ G_2 &= 2 \end{aligned}$$

All three tests for fathoming this sub-problem are false so we create sub- problems 8 and 9 by branching on G_1. The queue is now:

Sub-problem Number	Parent	New Constraint	OFV (Parent)
6	3	$G_2 = 0$	63.489
7	3	$G_2 \geq 1$	63.489
8	5	$G_1 \leq 1$	62.559
9	5	$G_1 \geq 2$	62.559

The largest number in the final column gives UB = 63.489, and hence

$$62.5 \leq \text{OFV}^* \leq 63.489$$

Solving sub-problem 6 yields:

$$\begin{aligned}
\text{OFV}(6) &= 63.243 \\
X &= 0.000000 \\
Y &= 0.238095 \\
G_1 &= 7.452381 \\
G_2 &= 0
\end{aligned}$$

None of the three fathoming tests applies, so we branch on variable G_1 to create sub-problems 10 and 11. The queue is:

Sub-problem Number	Parent	New Constraint	OFV (Parent)
7	3	$G_2 \geq 1$	63.489
8	5	$G_1 \leq 1$	62.559
9	5	$G_1 \geq 2$	62.559
10	6	$G_1 \leq 7$	63.243
11	6	$G_1 \geq 8$	63.243

Removing sub-problem 7 from the queue we see that it is infeasible, so we do not branch on it. With 7 gone the upper bound becomes 63.243. Next, we remove sub-problem 10, and solve it to obtain:

$$\begin{aligned}
\text{OFV}(10) &= 62.819 \\
X &= 0.1875 \\
Y &= 0.1250 \\
G_1 &= 7 \\
G_2 &= 0
\end{aligned}$$

Examining this we see that we must branch on Y to create sub-problems 12 and 13. The queue is:

Sub-problem Number	Parent	New Constraint	OFV (Parent)
8	5	$G_1 \leq 1$	62.559
9	5	$G_1 \geq 2$	62.559
11	6	$G_1 \geq 8$	63.243
12	10	$Y = 0$	62.819
13	10	$Y = 1$	62.819

We remove sub-problem 11 from the queue, and find that it is infeasible. With it now gone from the queue, the upper bound becomes 62.819. Examining sub-problem 12 we find:

$$\begin{aligned} \mathrm{OFV}(12) &= 62.675 \\ X &= 1.25 \\ Y &= 0 \\ G_1 &= 7 \\ G_2 &= 0 \end{aligned}$$

This is integer so we do not branch on it. Furthermore, it is better than the incumbent so this solution becomes the new incumbent. Hence we have established a new *lower* bound of 62.675, and it follows that:

$$62.675 \leq \mathrm{OFV}^* \leq 63.243$$

At this point we can see that even if sub-problems 8 and 9 are feasible, they cannot be optimal since the OFV cannot exceed the current lower bound. Hence sub-problems 8 and 9 are purged from the queue. This leaves only sub-problem 13, which we find to be infeasible. The queue is now empty, and we see that sub-problem 10 is optimal with $\mathrm{OFV}^* = 62.675$.

The optimal integer solution is

$$\begin{aligned} \mathrm{OFV}^* &= 62.675 \\ X^* &= 1.25 \\ Y^* &= 0 \\ G_1^* &= 7 \\ G_2^* &= 0 \end{aligned}$$

The convergence of the lower and upper bounds is shown in the following table, the figures referring to the *end* of the iteration.

Iteration	Sub-problem	LB	UB
1	1	56.300	63.846
2	2	56.300	63.846
3	3	56.300	63.571
4	4	62.500	63.571
5	5	62.500	63.488
6	6	62.500	63.488
7	7	62.500	63.243
8	10	62.500	63.243
9	11	62.500	62.819
10	12 (8,9)	62.675	62.819
11	13	62.675	62.675

The rules for choosing a variable on which to branch, and the rule for determining the order in which the sub-problems are solved are not cast in stone. For example, a simpler rule is to solve the sub-problems in the order in which they were created. If we had done this, the eighth and ninth sub-problems would have had to have been solved. The solution to the eighth sub-problem is:

$$\begin{aligned}
\mathrm{OFV}(8) &= 59.271 \\
X &= 1.764706 \\
Y &= 0 \\
G_1 &= 5 \\
G_2 &= 2
\end{aligned}$$

Because $59.271 < \mathrm{LB} = 62.5$, we would not have branched further down this path. Sub-problem 9 is in fact infeasible. Hence the simpler rule would have required 13 iterations rather than the 11 which we found earlier.

C.2.3 A Minimization Example

The complete formulation of the fixed charge problem discussed earlier is:

$$
\begin{array}{lrcl}
\text{minimize} & 2000Y_1+3000Y_2+1500Y_3+2400Y_4+2700Y_5 & & \\
& +3.8R_1+2.9R_2+4.2R_3+3.4R_4+3.6R_5 & & \\
& +4.6O_1+4.1O_2+5.6O_3+4.2O_4+5.1O_5 & & \\
\text{subject to} & & & \\
(1) & R_1-1000Y_1 & \leq & 0 \\
(2) & R_2-1200Y_2 & \leq & 0 \\
(3) & R_3-1500Y_3 & \leq & 0 \\
(4) & R_4-1300Y_4 & \leq & 0 \\
(5) & R_5-1400Y_5 & \leq & 0 \\
(6) & O_1-400Y_1 & \leq & 0 \\
(7) & O_2-550Y_2 & \leq & 0 \\
(8) & O_3-600Y_3 & \leq & 0 \\
(9) & O_4-450Y_4 & \leq & 0 \\
(10) & O_5-500Y_5 & \leq & 0 \\
(11) & R_1+R_2+R_3+R_4+R_5+ & & \\
& O_1+O_2+O_3+O_4+O_5 & \geq & 5100
\end{array}
$$

$$Y_i \in \{0,1\} \quad R_i, O_i \geq 0 \quad i=1,\ldots,5$$

We begin by relaxing the integrality restrictions, replacing $Y_i \in \{0,1\}$ by $0 \leq Y_i \leq 1$ for $i=1,\ldots,5$. The relaxed model is not integer; omitting the continuous variables the solution is:

$$
\begin{array}{rcl}
\text{OFV}(1) & = & 25,786.58 \\
Y_1 & = & 0 \\
Y_2 & = & 1 \\
Y_3 & = & 1 \\
Y_4 & = & 1 \\
Y_5 & = & 0.052632
\end{array}
$$

Since this solution is not naturally integer, we need to perform the branch and bound algorithm, which is shown in Figure C.2.

Because this is a min model, OFV(1) will serve as a lower bound to the OFV of the optimal integer solution. To obtain an upper bound, we can round Y_5 up to

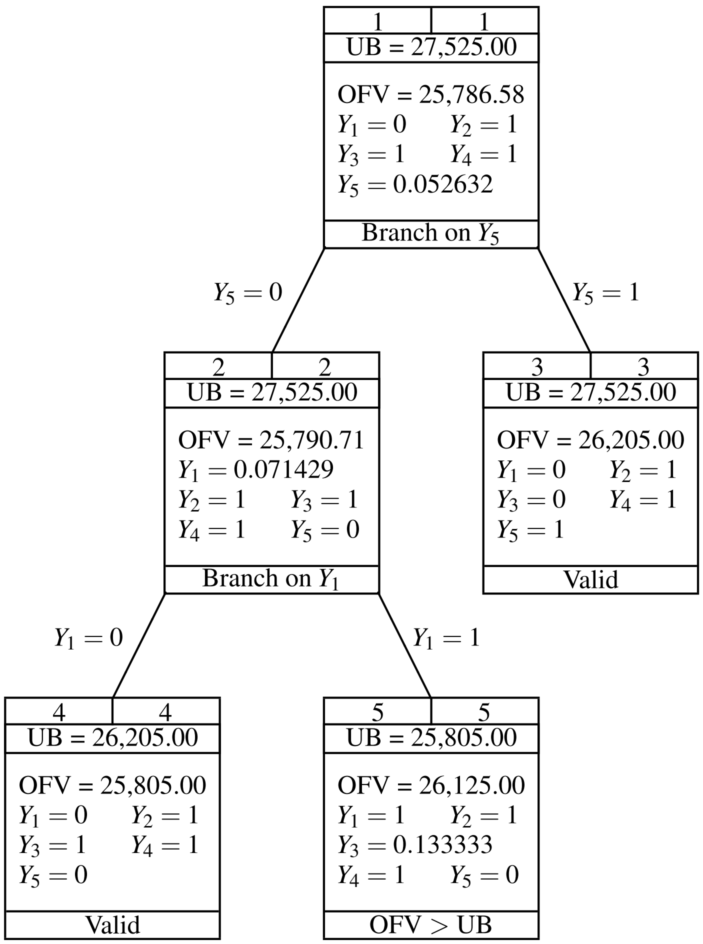


Figure C.2: Branch and Bound Tree for a Minimization Example

1, keeping the other Y variables at their current values. Solving for the continuous variables we obtain an OFV of 27,525.00. Hence

$$25,786.58 \leq \text{OFV}^* \leq 27,525.00$$

We branch on Y_5, adding constraint $Y_5 = 0$ to sub-problem 2, and adding constraint $Y_5 = 1$ to sub-problem 3. Solving sub-problem 2 we obtain:

$$\begin{aligned} \text{OFV}(2) &= 25,790.10 \\ Y_1 &= 0.071429 \\ Y_2 &= 1 \\ Y_3 &= 1 \\ Y_4 &= 1 \\ Y_5 &= 0 \end{aligned}$$

Since none of the three fathoming tests applies, we create sub-problems 4 and 5, adding $Y_1 = 0$ to the former, and $Y_1 = 1$ to the latter. Removing sub-problem 3 from the queue and solving it we obtain:

$$\begin{aligned} \text{OFV}(3) &= 26,205.00 \\ Y_1 &= 0 \\ Y_2 &= 1 \\ Y_3 &= 0 \\ Y_4 &= 1 \\ Y_5 &= 1 \end{aligned}$$

Since this solution is integer, we do not branch on it. Furthermore, this integer solution is better than the incumbent, so it becomes the new incumbent with UB = 26,205.00. We also have a new lower bound, which is 25,790.71. The search has now been narrowed to

$$25,790.71 \leq \text{OFV}^* \leq 26,205.00$$

We remove sub-problem 4 from the queue and solve it:

$$\begin{aligned} \text{OFV}(4) &= 25,805.00 \\ Y_1 &= 0 \end{aligned}$$

$$\begin{aligned} Y_2 &= 1 \\ Y_3 &= 1 \\ Y_4 &= 1 \\ Y_5 &= 0 \end{aligned}$$

This integer solution becomes the new incumbent with UB = 25,805.00, giving:

$$25,790.71 \leq \text{OFV}^* \leq 25,805.00$$

Solving sub-problem 5 we determine:

$$\begin{aligned} \text{OFV}(5) &= 26,125.00 \\ Y_1 &= 0 \\ Y_2 &= 1 \\ Y_3 &= 0.133333 \\ Y_4 &= 1 \\ Y_5 &= 0 \end{aligned}$$

This solution has OFV $>$ UB, so according to the second fathoming test, we do not proceed further down this branch. The queue is now empty, so the incumbent is optimal. The optimal solution for all variables is:

$$\begin{aligned} \text{OFV}^* &= 25,805.00 \\ Y_1^* &= 0 \\ Y_2^* &= 1 \\ Y_3^* &= 1 \\ Y_4^* &= 1 \\ Y_5^* &= 0 \\ R_1^* &= 0 \\ R_2^* &= 1200 \\ R_3^* &= 1500 \\ R_4^* &= 1300 \\ R_5^* &= 0 \\ O_1^* &= 0 \\ O_2^* &= 550 \end{aligned}$$

$$O_3^* = 100$$
$$O_4^* = 450$$
$$O_5^* = 0$$

The lower and upper bounds at the *end* of each iteration were:

Iteration	Sub-problem	LB	UB
1	1	25,786.58	27,525.00
2	2	25,786.58	27,525.00
3	3	25,790.71	26,205.00
4	4	25,790.71	25,805.00
5	5	25,805.00	25,805.00

C.2.4 Formal Statement of the Algorithm

STEP ONE

(i) Consider the original integer model. Create a new model in which we relax the integrality restrictions, by replacing them with the standard non-negativity restrictions, and in the case of a 0/1 variable, by adding a constraint requiring that this variable be less than or equal to 1. This relaxed model is called sub-problem 1. Solve sub-problem 1 as a linear optimization problem.
(ii) Should there be no feasible solution to sub-problem 1, then there is no feasible solution to the original model, and hence STOP.
(iii) Let the OFV of sub-problem 1 be denoted as OFV(1). If the solution to sub-problem 1 obeys the integrality restrictions of the original model, then this solution is optimal with OFV* = OFV(1), and hence STOP.
(iv) For a max model, let UB = OFV(1); for a min model, let LB = OFV(1).
(v) Let $m = 1$, $M = 1$, and $n = 1$. [m is the number of the current sub- problem, M is the number of sub-problems created so far, and n is the iteration number.]

STEP TWO

(i) Try to find a feasible solution to the original integer model. If one is found, go to (ii), otherwise go to (iii).
(ii) Call this solution the *incumbent* solution. For a max model, let LB = OFV(incumbent); for a min model, let UB = OFV(incumbent). Go to (iv).

(iii) For a max model, let LB = $-\infty$; for a min model, let UB = $+\infty$.
(iv) Hence we have, for max or min,

$$\text{LB} \leq \text{OFV}^* \leq \text{UB}$$

STEP THREE

(i) Choose a variable which is required to be integer in the original model, but whose value is not integer in the linear solution to sub-problem m. [A possible strategy here is to choose the one whose current value has a fractional component closest to 0.5]
(ii) Create two sub-problems, numbered $M+1$ and $M+2$, and increase the value of M by 2.
(iii) In sub-problem $M+1$, the model is as it was in sub-problem m except that we add a constraint which requires that the variable chosen in (i) be less than or equal to the integer number which is just below its current value.
(iv) In sub-problem $M+2$, the model is as it was in sub-problem m except that we add a constraint which requires that the variable chosen in (i) be greater than or equal to the integer number which is just above its current value.
(v) Add both sub-problems to a queue of sub-problems to be examined.

STEP FOUR

(i) If the queue is *not* empty, then go to (iv).
(ii) If there be no incumbent, then go to (iii). Otherwise, the incumbent is optimal. If max, then UB = LB, and OFV* = LB, and if min, then LB = UB, and OFV* = UB, and hence STOP.
(iii) Since no incumbent exists, the original model has no feasible solution, and hence STOP.
(iv) For a max model, UB = largest OFV(parent). For a min model, LB = smallest OFV(parent).
(v) Remove a sub-problem from the queue. [A possible strategy here is choose the sub-problem whose parent has the largest OFV for a max problem or smallest OFV for a min problem. Ties can be broken using FIFO – First In, First Out.] This sub-problem has number m.
(vi) Increase the value of n by one.

STEP FIVE

We are at iteration n of the branch and bound algorithm.
(i) Solve sub-problem m.
(ii) If the solution is infeasible, then go to STEP FOUR. [Sub-problem m has been fathomed at iteration n.]
(iii) If, for a max model, OFV $\leq$ LB, or if, for a min model, OFV $\geq$ UB, then go to STEP FOUR. [Sub-problem m has been fathomed at iteration n.]
(iv) If the solution does *not* obey the integrality restrictions of the original model, then go to STEP THREE.
(v) If the integer solution to sub-problem m has OFV$(m) \leq$ LB for a max model, or OFV$(m) \geq$ UB for a min model, then go to STEP FOUR. [If true, sub-problem m is no better than the incumbent.]
(vi) Sub-problem m becomes the new incumbent.
For a max model, LB = OFV(m)
For a min model, UB = OFV(m)
(vii) Purge each sub-problem in the queue whose parent's OFV is less than LB for a max problem, or greater than UB for a min problem.
(viii) Go to STEP FOUR.

C.2.5 Manual Implementation

There are two ways for a student to use the branch and bound algorithm. One way is to use it to solve two-variable examples, solving each sub-problem graphically. For example, for the graphical model presented earlier, there is a tie for the variable on which to branch, since $X_1 = 4\frac{2}{7}$, and $X_2 = 3\frac{5}{7}$, and hence both are $\frac{2}{7}$ from the nearest integer. Arbitrarily choosing X_1, we let $X_1 \leq 4$ in sub-problem 2, and $X_1 \geq 5$ in sub-problem 3, we obtain $X_1 = 4$, and $X_2 = 3\frac{2}{3}$ in sub-problem 2, and sub-problem 3 is infeasible. Descending from sub-problem 2 we create sub-problem 4 with $X_2 \leq 3$, and sub-problem 5 with $X_2 \geq 4$. In sub-problem 4, $X_1 = 2.5$, and $X_2 = 3$; sub-problem 5 is infeasible. Descending from sub-problem 4, we create sub-problem 6 with $X_1 \leq 2$, and sub-problem 7 with $X_1 \geq 3$. Sub-problem 6 is integer with $X_1 = 2$, $X_2 = 3$, and OFV = 17. This becomes the incumbent solution. Sub-problem 7 is infeasible. There being no sub-problems left to examine, the incumbent found at sub-problem 6 is optimal. (Of course, a two-variable example can be solved graphically directly).

The other way is to operate a master problem by hand, solving each sub-problem using a linear optimization software package. The only point of doing either of these things is to study the branch and bound algorithm for illustrative purposes. Obviously, in practice integer models are solved directly using a spreadsheet solver or a dedicated optimization package.

C.2.6 Problems for Student Completion

1. Solve the following integer optimization model using the branch and bound algorithm. At each iteration of the master problem solve the relaxed problem using the graphical solution technique for linear optimization. Use the branching rules given in the text.

$$\begin{array}{rrcrcl}
\max & X_1 & + & 2X_2 & & \\
\text{subject to} & & & & & \\
(1) & 2X_1 & + & X_2 & \leq & 5 \\
(2) & X_1 & + & 4X_2 & \leq & 12 \\
(3) & 5X_1 & + & X_2 & \geq & 5 \\
 & & & & & \\
 & X_1 & , & X_2 & \in & \{0,1,2,\ldots\}
\end{array}$$

2. Solve the following integer optimization model using the branch and bound algorithm. At each iteration of the master problem solve the relaxed problem using the graphical solution technique for linear optimization. Use the branching rules given in the text.

$$\begin{array}{rrcrcl}
\min & 4X_1 & + & 3X_2 & & \\
\text{subject to} & & & & & \\
(1) & X_1 & + & 2X_2 & \geq & 4 \\
(2) & 5X_1 & + & 2X_2 & \geq & 10 \\
(3) & 5X_1 & + & 3X_2 & \leq & 15 \\
 & & & & & \\
 & X_1 & , & X_2 & \in & \{0,1,2,\ldots\}
\end{array}$$

3. Solve the following integer optimization model using the branch and bound algorithm. At each iteration of the master problem solve the relaxed problem using the graphical solution technique for linear optimization. Use the branching rules given in the text of this chapter.

   Note that a trivial feasible solution to the original problem is $X_1 = 0$, and $X_2 = 0$.

$$
\begin{array}{rrcrcrcr}
\max & 5X_1 & + & 6X_2 & & \\
\text{subject to} & & & & & \\
(1) & 20X_1 & + & 5X_2 & \leq & 64 \\
(2) & 8X_1 & + & 10X_2 & \leq & 41 \\
& & & & & \\
& X_1 & , & X_2 & \in & \{0,1,2,\ldots\}
\end{array}
$$

4. Solve the following integer optimization model using the branch and bound algorithm. At each iteration of the master problem solve the relaxed problem using software for *linear* optimization (i.e. do not declare the variables to be integer). Use the branching rules given in the text.

$$
\begin{array}{rrcrcrcr}
\max & 4X_1 & + & 3X_2 & + & 10X_3 & & \\
\text{subject to} & & & & & & & \\
(1) & 3X_1 & + & 2X_2 & + & 8X_3 & \leq & 37 \\
(2) & 2X_1 & + & 5X_2 & + & 4X_3 & \leq & 25 \\
(3) & 7X_1 & + & 4X_2 & + & 6X_3 & \leq & 48 \\
(4) & 5X_1 & + & X_2 & + & 2X_3 & \geq & 23 \\
& & & & & & & \\
& X_1 & , & X_2 & , & X_3 & \in & \{0,1,2,\ldots\}
\end{array}
$$

5. Solve the following integer optimization model using the branch and bound algorithm. At each iteration of the master problem solve the relaxed problem using software for *linear* optimization (i.e. do not declare the variables to be integer). Use the branching rules given in the text.

$$
\begin{array}{rrcrcrcr}
\min & 7X_1 & + & 3X_2 & + & 2X_3 & & \\
\text{subject to} & & & & & & & \\
(1) & 8X_1 & + & 5X_2 & + & 4X_3 & \geq & 21 \\
(2) & 4X_1 & + & 2X_2 & + & 7X_3 & \geq & 18 \\
(3) & 6X_1 & + & 3X_2 & + & 2X_3 & \geq & 33 \\
(4) & 7X_1 & + & 6X_2 & + & 4X_3 & \leq & 57 \\
& & & & & & & \\
& X_1 & , & X_2 & , & X_3 & \in & \{0,1,2,\ldots\}
\end{array}
$$

Appendix D

Review of Differential Calculus

D.1 Overview

The reader will have presumably completed a course in differential calculus, in which an unconstrained function of a single variable is optimized. The process is:

1. Model the problem using a single variable x,[1] to create a function $f(x)$ that we seek to optimize (i.e., maximize or minimize depending on the situation).

2. Using the rules of differentiation, find the first derivative $f'(x)$.

3. Set $f'(x) = 0$, and solve this to obtain solution $\bar{x}$.

4. Find the second derivative $f''(x)$.

5. Evaluate $f''(x)$ at $x = \bar{x}$. If $f''(\bar{x}) > 0$, then the function has a local minimum at $x = \bar{x}$. If $f''(\bar{x}) < 0$, then the function has a local maximum at $x = \bar{x}$. If $f''(\bar{x}) = 0$, then further testing is required to determine whether this point is a local maximum, a local minimum, or neither of these.

[1]Though we have been using capital letters for variable names, here we use small x to reflect the using of most calculus textbooks.

D.2 Details of the Procedure

D.2.1 Rules of Differentiation

Some of the rules of differentiation mentioned in Step 2 are as follows:

	Rule	**Function**	**Derivative**
1		$f(x) = a$	$f'(x) = 0$
2	Power	$f(x) = x^n$	$f'(x) = nx^{n-1}$
3		$f(x) = ag(x)$	$f'(x) = ag'(x)$
$4(a)$		$f(x) = u(x) + v(x)$	$f'(x) = u'(x) + v'(x)$
$4(b)$		$f(x) = u(x) - v(x)$	$f'(x) = u'(x) - v'(x)$
5	Product	$f(x) = u(x)v(x)$	$f'(x) = u'(x)v(x) + u(x)v'(x)$
6	Quotient	$f(x) = \dfrac{u(x)}{v(x)}$	$f'(x) = \dfrac{u'(x)v(x) - u(x)v'(x)}{(v(x))^2}$
7		$f(x) = e^{ax}$ Special case $a = 1$ $f(x) = e^x$	$f'(x) = ae^{ax}$ $f'(x) = e^x$
8		$f(x) = \ln(ax)$	$f'(x) = \dfrac{1}{x}$ $(a > 0; x > 0)$
9	Chain	$f(u)$, where $u = u(x)$	$f'(x) = f'(u)u'(x)$

D.2.2 Finding Extreme Points

When we ***optimize*** a function $f(x)$, we are trying to find the value or values of x at which the function is maximized or minimized. A point $\bar{x}$ at which a maximum or minimum of $f(x)$ occurs is called an ***extreme point***.

Suppose that we seek the maximum of a function. To be precise, the maximum of the function over the domain of the variable is called the ***global maximum*** (or ***absolute maximum***). This is what we seek, but we might first have to examine several local maxima. A ***local maximum*** (or ***relative maximum***) is a point which is higher than the neighbouring values of the function, but might not be a global maximum. However, if there is only one local maximum, then it must also be a global maximum. Fortunately, many functions have only one local maximum.

The same concepts apply if we are seeking to minimize a function. We want the ***global minimum***, which may be one of several local minima. However, if there is only one ***local minimum***, then it must also be a global minimum. Fortunately, many functions have only one local minimum.

Having a unique maximum/minimum is especially true of functions which arise from business applications, as opposed to contrived mathematical examples. We begin by considering an ***unconstrained*** function $f(x)$, which is continuous and differentiable for all real numbers.

D.2.3 Local Maxima and Minima

A ***necessary condition*** for a function to attain a local maximum or a local minimum at a point $\bar{x}$ is that

$$f'(\bar{x}) = 0$$

Another way of saying this is that

$$f(x) \text{ has a local max or min at } \bar{x} \Longrightarrow f'(\bar{x}) = 0$$

The converse, however, is not true. For example, if $f(x) = (x-5)^3$, then $f'(x) = 3(x-5)^2$, which is 0 when x is 5. However, the function is neither maximized nor minimized at $x = 5$.

A point x is said to be ***stationary*** if $f'(x) = 0$. Hence the statement "find all the stationary points of $f(x)$" means "find all the values of x for which $f'(x) = 0$". At a stationary point, the function could attain

- a local maximum or
- a local minimum or
- neither a local maximum nor a local minimum

D.2.4 The Second Derivative Test

To help determine which of the three cases applies, we need to examine the second derivative at the stationary point $x = \bar{x}$. If the second derivative is negative ($f''(\bar{x}) < 0$), then the function has a local maximum at the stationary point. If the second derivative is positive ($f''(\bar{x}) > 0$), then the function has a local minimum at the stationary point. If the second derivative is zero ($f''(\bar{x}) = 0$), then we do not know what we have: there could be a local maximum, or there could be a local minimum, or there could be neither one nor the other.[2] To determine what we have, we must examine the third or possibly even higher order derivatives at the stationary point. To do this, find the smallest value of n such that

$$f^{(n)}(\bar{x}) \neq 0$$

If n is odd, then the function attains neither a maximum nor a minimum at $\bar{x}$. If n is even, then

(i) there is a local maximum if $f^{(n)}(\bar{x}) < 0$;
(ii) there is a local minimum if $f^{(n)}(\bar{x}) > 0$.

D.3 Examples

In the following examples, we seek to discover all stationary points, and to determine whether at each the function attains a local maximum, or a local minimum, or neither.

D.3.1 Example 1

$$f(x) = 3x^2 - 9x + 5$$

The first derivative is

$$f'(x) = 6x - 9$$

At $f'(x) = 0$,

$$\begin{aligned} 6x - 9 &= 0 \\ 6x &= 9 \\ x &= 1.5 \end{aligned}$$

[2] A point where the sign of the second derivative changes is called a "point of inflection". While the second derivative must be 0 at an inflection point, the converse is not true: e.g. $f(x) = x^4$ has $f''(0) = 0$, but $x = 0$ is not an inflection point because the sign of the second derivative does not change. Note also that an inflection point need not be a stationary point.

So the function $f(x) = 3x^2 - 9x + 5$ has a single stationary point at $x = 1.5$. The value of the function at this point is

$$\begin{aligned} f(1.5) &= 3(1.5^2) - 9(1.5) + 5 \\ &= 6.75 - 13.5 + 5 \\ &= -1.75 \end{aligned}$$

The second derivative is

$$f''(x) = 6$$

Therefore, $f''(1.5) = 6 > 0$. Hence $f(x)$ has a local minimum at $x = 1.5$. Note that $f(x) \geq -1.75$ for all values of x.

D.3.2 Example 2

$$f(x) = -x^2 + 8x + 15$$

Hence $f'(x) = -2x + 8$. At $f'(x) = 0$, $-2x + 8 = 0$, and therefore the solitary stationary point of $f(x)$ occurs at $x = 4$. The value of the function at this point is

$$\begin{aligned} f(4) &= -4^2 + 8(4) + 15 \\ &= -16 + 32 + 15 \\ &= 31 \end{aligned}$$

The second derivative is $f''(x) = -2$, hence $f''(4) = -2 < 0$. A local maximum is obtained at $x = 4$.

D.3.3 Example 3

$$f(x) = \frac{x}{e^x}$$

First, we solve this as written. Using the quotient rule we obtain

$$f'(x) = \frac{e^x(1) - xe^x}{(e^x)^2}$$

Factoring e^x from the numerator and the denominator gives

$$f'(x) = \frac{1 - x}{e^x}$$

The first derivative is zero when the numerator is zero, i.e. $f'(x) = 0$ if and only if $1 - x = 0$, which occurs at $x = 1$; this is the only stationary point. The value of the function at this point is $f(1) = e^{-1} \approx 0.3679$.

To find the second derivative we again use the quotient rule, with $u(x) = 1 - x$ and $v(x) = e^x$.

$$f''(x) = \frac{e^x(-1) - (1-x)e^x}{(e^x)^2}$$

After factoring e^x and simplifying we obtain

$$f''(x) = \frac{x-2}{e^x}$$

At the stationary point $x = 1$ the value of the second derivative is

$$\begin{aligned} f''(1) &= \frac{1-2}{e^1} \\ &= \frac{-1}{e} \\ &< 0 \end{aligned}$$

Hence, at $x = 1$, the function $f(x) = \dfrac{x}{e^x}$ attains a local maximum.

Alternate Solution Here is another way of solving this problem, which is somewhat easier. We re-write $f(x)$ as

$$f(x) = xe^{-x}$$

Using product rule we obtain

$$f'(x) = e^{-x} + x(-1)e^{-x}$$

which simplifies to

$$f'(x) = (1-x)e^{-x}$$

When $f'(x) = 0$, we see as before that $x = 1$. To find the second derivative we use product rule with $u(x) = 1 - x$ and $v(x) = e^{-x}$.

$$f''(x) = (-1)e^{-x} + (1-x)(-1)e^{-x}$$

which simplifies to

$$f''(x) = (x-2)e^{-x}$$

As before, the second derivative is negative at $x = 1$, and so we have found a local maximum.

D.3.4 Example 4

$$f(x) = (x-2)^5$$

Hence $f'(x) = 5(x-2)^4$. At $f'(x) = 0$, $5(x-2)^4 = 0$, hence there is a single stationary point at $x = 2$. The value of the function at this point is $f(x = 2) = (2-2)^5 = 0$.

The second derivative is $f''(x) = 20(x-2)^3$, so at the stationary point $f''(2) = 20(2-2)^3 = 0$. The second order test is therefore inconclusive, hence we find the higher derivatives. These are:

$$\begin{aligned}
f^{(3)}(x) &= 60(x-2)^2 \\
\text{Therefore } f^{(3)}(2) &= 60(2-2)^2 = 0 \\
\\
f^{(4)}(x) &= 120(x-2) \\
\text{Therefore } f^{(4)}(2) &= 120(2-2) = 0 \\
\\
f^{(5)}(x) &= 120 \\
\text{Therefore } f^{(5)}(2) &= 120 \neq 0
\end{aligned}$$

Hence we have $n = 5$, which is odd, and therefore the function attains neither a local maximum nor a local minimum at the stationary point.

D.3.5 Example 5

$$f(x) = x^3 - 5x^2 + 7x + 8$$

Therefore

$$f'(x) = 3x^2 - 10x + 7$$

To find where $f'(x) = 0$ we need to solve

$$3x^2 - 10x + 7 = 0$$

This is a quadratic equation with $a = 3$, $b = -10$, and $c = 7$.[3] Using the quadratic formula we obtain

$$\begin{aligned} x &= \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(7)}}{2(3)} \\ &= \frac{10 \pm 4}{6} \end{aligned}$$

Hence the two roots are $x = 1$ and $x = 2\frac{1}{3}$; $f(x)$ has stationary points at $x = 1$ and $x = 2\frac{1}{3}$ (or $\frac{7}{3}$). The values of the function at these two points are:

$$\begin{aligned} f(x=1) &= 1^3 - 5(1^2) + 7(1) + 8 \\ &= 1 - 5 + 7 + 8 \\ &= 11 \end{aligned}$$

and

$$\begin{aligned} f(x=\tfrac{7}{3}) &= \left(\frac{7}{3}\right)^3 - 5\left(\frac{7}{3}\right)^2 + 7\left(\frac{7}{3}\right) + 8 \\ &= 12.7037... - 27.2222... + 16.3333... + 8 \\ &\approx 9.8148 \qquad\qquad (\text{or } 9\tfrac{22}{27} \text{ exactly}) \end{aligned}$$

The second derivative is $f''(x) = 6x - 10$. At $x = 1$ we have

$$\begin{aligned} f''(x=1) &= 6(1) - 10 \\ &= -4 \\ &< 0 \end{aligned}$$

Hence $f(x)$ has a local maximum at $x = 1$. At $x = 2\frac{1}{3} = \frac{7}{3}$ the second derivative is

$$\begin{aligned} f''(x=\tfrac{7}{3}) &= 6\left(\frac{7}{3}\right) - 10 \\ &= 14 - 10 \\ &= 4 \\ &> 0 \end{aligned}$$

Hence $f(x)$ has a local minimum at $x = 2\frac{1}{3}$.

---

[3] Recall that a quadratic equation is one of the form $ax^2 + bx + c = 0$ with $a \neq 0$. As long as $b^2 \geq 4ac$ it will have real roots given by the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

D.3.6 Example 6

$$f(x) = \frac{e^x}{x} \qquad (x > 0)$$

Using the quotient rule we obtain

$$\begin{aligned} f'(x) &= \frac{xe^x - e^x(1)}{x^2} \\ &= \frac{e^x(x-1)}{x^2} \end{aligned}$$

The first derivative is zero when the numerator $e^x(x-1) = 0$. Since $e^x > 0$ for all x, the stationary point occurs when $x - 1 = 0$, i.e. at $x = 1$. The value of the function at this point is

$$f(1) = \frac{e^1}{1} = e \qquad (\approx 2.718)$$

To find the second derivative of $f(x)$ we use the quotient rule letting $u(x) = e^x(x-1)$ and $v(x) = x^2$. We use the product rule to find $u'(x)$:

$$\begin{aligned} u'(x) &= e^x(1) + e^x(x-1) \\ &= xe^x \end{aligned}$$

The derivative of $v(x)$ is simply $2x$. Hence

$$\begin{aligned} f''(x) &= \frac{x^2(xe^x) - e^x(x-1)2x}{(x^2)^2} \\ &= \frac{e^x(x^3 - (x-1)2x)}{x^4} \\ &= \frac{e^x(x^2 - (x-1)2)}{x^3} \\ &= \frac{e^x(x^2 - 2x + 2)}{x^3} \end{aligned}$$

At the stationary point $x = 1$,

$$\begin{aligned} f''(x=1) &= \frac{e^1(1^2 - 2(1) + 2)}{1^3} \\ &= e(1 - 2 + 2) \\ &= e \qquad (\approx 2.718) \\ &> 0 \end{aligned}$$

Therefore $f(x)$ has a local minimum at $x = 1$.

D.4 Global Maximum and Minimum

As we have said, if an unconstrained function has only one maximum/minimum, then that point must also be a global maximum/minimum. If a function has several local maxima/minima, the situation is more complicated.

One possibility is that a function could have local maxima/minima, but no global maximum/minimum. An example of this is the function given in example 5 above ($f(x) = x^3 - 5x^2 + 7x + 8$), which had a local maximum at $x = 1$, and a local minimum at $x = 2\frac{1}{3}$. Clearly as $x \to \infty$, $f(x)$ increases indefinitely, and hence $f(x)$ has no global maximum. As $x \to -\infty$, $f(x)$ decreases indefinitely, and hence $f(x)$ has no global minimum.

Another possibility is that the function has several local maxima and minima, and the global maximum/minimum is one of these. We would have to find all these points, and calculate the value of the function at each of these points.

D.4.1 Constrained Optimization

In this section we consider a function $f(x)$ which is continuous and differentiable over its domain. The domain is one of three forms:

(1) there is a lower ***endpoint*** a such that $a \leq x$, or

(2) there is an upper endpoint b such that $x \leq b$, or

(3) there are both lower and upper endpoints such that $a \leq x \leq b$.

Now, the endpoint(s) a and/or b are potential points of optimality. If both endpoints exist i.e. $a \leq x \leq b$, then $f(x)$ will have a maximum and a minimum somewhere. We solve such problems by first finding the stationary points (if any) which lie within the domain of the function.

If there is no stationary point within the domain, then either a is the value of x which maximizes $f(x)$, and b is the value of x which minimizes $f(x)$, or else $f(x)$ is minimized at $x = a$ and maximized at $x = b$. All we have to do is evaluate $f(a)$, $f(b)$, and compare them.

If $f(x)$ has only one stationary point $\bar{x}$ within the domain, i.e. $a \leq \bar{x} \leq b$, then the potential points for maximization or minimization are $x = a$, $x = \bar{x}$, and $x = b$. What we need to do is evaluate and compare $f(a)$, $f(\bar{x})$, and $f(b)$.

If there are two or more stationary points within the domain, then we would compare $f(x)$ evaluated at $x = a$ and $x = b$ (the endpoints) with $f(x)$ evaluated at each of the stationary points.

D.5 Examples

D.5.1 Example 1

We are given

$$f(x) = x^2 - 10x + 25$$

where the domain of $f(x)$ is $1 \leq x \leq 4$. We wish to find the value of x which minimizes $f(x)$.

The first derivative is

$$f'(x) = 2x - 10$$

At $f'(x) = 0$,

$$\begin{aligned} 2x - 10 &= 0 \\ 2x &= 10 \\ x &= 5 \end{aligned}$$

The sole stationary point $\bar{x} = 5$ is outside of the domain, which is $1 \leq x \leq 4$. It is easy to see that for this example the function decreases over the domain, and hence the minimum must occur at the higher endpoint, $x = 4$. Another thing we could do is evaluate the function at the two endpoints, and then see numerically where the function is minimized.

$$\begin{aligned} f(x = 1) &= 1^2 - 10(1) + 25 \\ &= 1 - 10 + 25 \\ &= 16 \end{aligned}$$

$$\begin{aligned} f(x = 4) &= 4^2 - 10(4) + 25 \\ &= 16 - 40 + 25 \\ &= 1 \end{aligned}$$

This function is minimized at $x = 4$ (and is maximized at $x = 1$).

D.5.2 Example 2

This is a constrained version of the earlier Example 5. We are given

$$f(x) = x^3 - 5x^2 + 7x + 8$$

where the domain of $f(x)$ is $0 \le x \le 2.75$.

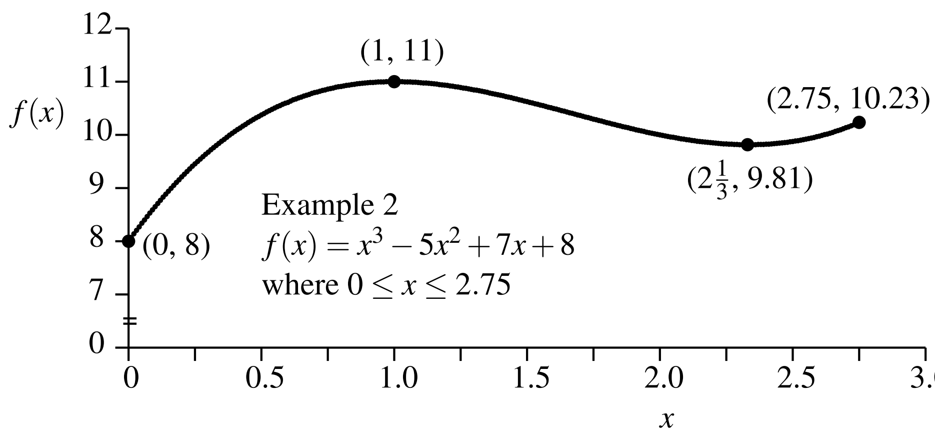
We found earlier that there is a local maximum at $x = 1$, and a local minimum at $x = 2\frac{1}{3}$. Both of these stationary points are within the domain. We need to compare $f(x)$ at the endpoints with $f(x)$ at the stationary points. Doing this yields:

$$
\begin{array}{llrcl}
 & x & f(x) & = & x^3 - 5x^2 + 7x + 8 \\
\\
\text{endpoint} & 0 & f(0) & = & 0^3 - 5(0)^2 + 7(0) + 8 \\
 & & & = & 8 \\
\\
\text{stationary point} & 1 & f(1) & = & 1^3 - 5(1)^2 + 7(1) + 8 \\
\text{(local maximum)} & & & = & 11 \\
\\
\text{stationary point} & 2\frac{1}{3} & f(\frac{7}{3}) & = & \left(\frac{7}{3}\right)^3 - 5\left(\frac{7}{3}\right)^2 + 7\left(\frac{7}{3}\right) + 8 \\
\text{(local minimum)} & & & = & 9.8148 \\
\\
\text{endpoint} & 2.75 & f(2.75) & = & (2.75)^3 - 5(2.75)^2 + 7(2.75) + 8 \\
 & & & = & 10.234
\end{array}
$$

The smallest value of $f(x)$ is 8, which occurs at $x = 0$. The largest value of $f(x)$ is 11, which occurs at $x = 1$. Therefore:

x	Type of extreme point
0	global minimum
1	global maximum
$2\frac{1}{3}$	local minimum
2.75	local maximum

Although not essential, it is often useful to graph the function over its domain. We already know four points on the graph: the two endpoints, and the two stationary points. Even with just these, we can make a rough sketch of the graph. By calculating a few more points, we can obtain a more accurate graph. Doing this we obtain:


12
11
10
9
8
7
0
f(x)
(1, 11)
(2.75, 10.23)
(2 1/3, 9.81)
(0, 8)
Example 2
$f(x) = x^3 - 5x^2 + 7x + 8$
where $0 \leq x \leq 2.75$
0
0.5
1.0
1.5
2.0
2.5
3.0
x

Appendix E

Decision Analysis Extensions

E.1 Probability

E.1.1 Some Preliminaries

Before we consider the subject of probability, we first mention three objects which will help us understand probability.

Coin

A coin has two essentially flat sides. Since it is the custom to have someone's portrait on one side of the coin, we refer to this side of the coin as "heads", and we refer to the other side as "tails". Although it may be theoretically possible for a coin, when flipped in the air, to land on its cylindrical edge, we generally discount this possibility, and say that when a coin is flipped there are two possible outcomes: either it lands with the "heads" side up, or it lands with the "tails" side up, and we refer to these two outcomes as "heads" and "tails" respectively.

Die

A cube is a six-sided object, each side being a square. A die is a cube in which each side has a different number of dots ranging from 1 to 6 inclusive. When a die is flipped it will land on one of its six sides. The outcomes of flipping a die are expressed by the various number of dots on the "up" side of the die. Hence there are six outcomes: 1, 2, 3, 4, 5, and 6 dots showing. The plural of die is *dice*.

Deck of Cards

Unless specified otherwise, we will consider the standard deck of 52 cards, which is comprised of 13 cards in each of four "suits". The four suits and their symbols are clubs $\clubsuit$, diamonds $\diamondsuit$, hearts $\heartsuit$, and spades $\spadesuit$. Though we cannot show the suits in colour, $\diamondsuit$ and $\heartsuit$ are the "red suits", and $\clubsuit$ and $\spadesuit$ are the "black suits". Within each suit, there are three "face cards": the King (K), the Queen (Q), and the Jack (J). There are also the "rank cards": 10, 9, 8, 7, 6, 5, 4, 3, 2. A thirteenth card, called the Ace (A), can rank either below the 2 or above the King, depending on the game. When we deal a card at random from a deck of 52 cards, there are 52 possible outcomes:

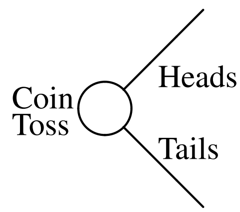
2$\clubsuit$	3$\clubsuit$	4$\clubsuit$	5$\clubsuit$	6$\clubsuit$	7$\clubsuit$	8$\clubsuit$	9$\clubsuit$	10$\clubsuit$	J$\clubsuit$	Q$\clubsuit$	K$\clubsuit$	A$\clubsuit$
2$\diamondsuit$	3$\diamondsuit$	4$\diamondsuit$	5$\diamondsuit$	6$\diamondsuit$	7$\diamondsuit$	8$\diamondsuit$	9$\diamondsuit$	10$\diamondsuit$	J$\diamondsuit$	Q$\diamondsuit$	K$\diamondsuit$	A$\diamondsuit$
2$\heartsuit$	3$\heartsuit$	4$\heartsuit$	5$\heartsuit$	6$\heartsuit$	7$\heartsuit$	8$\heartsuit$	9$\heartsuit$	10$\heartsuit$	J$\heartsuit$	Q$\heartsuit$	K$\heartsuit$	A$\heartsuit$
2$\spadesuit$	3$\spadesuit$	4$\spadesuit$	5$\spadesuit$	6$\spadesuit$	7$\spadesuit$	8$\spadesuit$	9$\spadesuit$	10$\spadesuit$	J$\spadesuit$	Q$\spadesuit$	K$\spadesuit$	A$\spadesuit$

A deck of cards often contains two other cards called the Jokers. Unless indicated to the contrary, we will not use these two cards.

E.1.2 Events and Outcomes

We will refer to situations which have uncertainty as ***events***. Some examples are (1) the flip of a coin, (2) the toss of a die, and (3) the dealing of a card. The first is an event with two possible ***outcomes***, of which exactly one will occur; in short we will say that this is an event with two outcomes. The toss of a die is an event with six outcomes, and the dealing of a card from a standard deck is an event with 52 outcomes. Mathematics textbooks refer to what we call an event as an "experiment", but this word is not very satisfactory in a business context, where the event may be the drilling of a hole in the search for oil.

We can draw a picture of an event with its outcomes. The event is drawn as a circle, and each outcome is a straight line (called a "branch") radiating from the right hemisphere of the circle. For the coin toss we have:



E.1.3 Interpretations of 'Probability'

What do we mean by the word "probability"? There is no single accepted definition. Instead there are three interpretations of the word, each of which has limited validity. All three have one thing in common, however: a probability is a number between 0 and 1 inclusive.

Laplacian Interpretation

Named after the mathematician Laplace, the interpretation assumes that all outcomes are equally likely. Since something has to happen when a card is dealt from a deck of 52 cards, we think of the total probability of all outcomes as 100%, or 1, and if it can be assumed that each outcome is equally likely, then the probability of obtaining a particular outcome, say the 8♣, is one chance in fifty-two, or $\frac{1}{52}$. This assumption is quite reasonable for items such as coins, dice, and cards, but it has limited applicability elsewhere. For example, suppose that we drill for oil at a particular spot. We might think of this event as having two outcomes: oil is found, or oil is not found. We know from the experience of many who have drilled for oil that we cannot think of the probabilities of these two outcomes in Laplacian terms – we are far more likely to not find oil.

Empirical Frequency Interpretation

Suppose that a company manufactures 500 units of an item, and then subjects each unit to a test, with 490 meeting the objectives of the test, and the other 10 units classified as being "defective". In this sample of 500, the probability (before testing, of course) that any one unit would turn out to be defective is $\frac{10}{500} = 0.02$. We often assume that things will keep going are they are now, so that if a 501st unit is produced, then its probability of being defective is also 0.02. Even if we can agree that no changes to the production process have occurred which might change the probability of being defective, we might (and should) still wonder if the sample size of 500 is large enough. If we had sampled 10000 units, would we have found 200 defective units? Another problem with this approach is that it only deals with the situation where a history has been observed. What does it mean, for example, to say that a rocket sent to Jupiter has 1% chance of crashing into the planet, if we have never sent a rocket to Jupiter before?

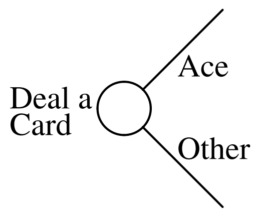
Subjective Interpretation

One other school of thought suggests that a probability is a number which reflects a particular individual's interpretation of the likelihood of a particular outcome. For example, a geologist may estimate that at a particular location, given his or her experience with similar areas, that the probability of there being an underground pool of oil is 0.02 (or 2%). The problem with this approach is that two people can come up with different probabilities, and neither person can prove that one number is better than the other.

When dealing with problems involving the tossing of a coin with its two outcomes, the rolling of a die with its six outcomes, or the drawing of a card at random from a standard deck with its 52 outcomes, we will use the Laplacian assumption. A more complicated probability, such as determining the probability of being dealt three Kings in a deal of five cards, can be derived using a formula. For other types of problems, such as the ones one generally sees in Business examples, we will use given subjective probabilities, or probabilities which can be derived from such subjective probabilities.

E.1.4 Grouping Elementary Outcomes

We are often interested in knowing the probability that an outcome from a group of outcomes will occur. For example, we deal a card from a shuffled deck of cards and we want to know the probability that the card is an Ace (of any suit). Rather than think of 52 elementary outcomes of this event, it simplifies matters considerably to group the four outcomes Ace $\clubsuit$, Ace $\diamondsuit$, Ace $\heartsuit$, and Ace $\spadesuit$ together as one compound outcome called "Ace", and to group the other 48 outcomes as "not an Ace" or "other". The picture for this event is



In some books, the word "event" is used in the sense of "compound outcomes". This of course is not our usage as the word "event" has already been defined. When the context is clear, we will generically refer to an "outcome" to mean either an "elementary outcome" or a "compound outcome".

The probabilities of the two compound outcomes are found by considering the proportion of the 52 elementary outcomes on which they are based. We use the notation P(outcome x) to mean the probability of outcome x occurring. Hence

$$\begin{aligned} \mathrm{P(Ace)} &= \frac{\text{4 cards which are Aces}}{\text{52 cards in total}} \\ &= \frac{1}{13} \\ \mathrm{P(other)} &= \frac{48}{52} \\ &= \frac{12}{13} \end{aligned}$$

Note that

$$\begin{aligned} \mathrm{P(Ace)} + \mathrm{P(other)} &= \frac{1}{13} + \frac{12}{13} \\ &= 1 \end{aligned}$$

In defining outcomes two rules must be followed.

1. The outcomes must be ***mutually exclusive***. By this, we mean that two or more outcomes cannot occur simultaneously. For example, it would be WRONG to define three outcomes of a card deal event as "black", "club", and "red", since "black" and "club" overlap.

2. The outcomes must be ***complete***. By this, we mean that no other outcome of the event could occur. For example, it would be WRONG to define three outcomes of a card deal event as $\clubsuit$, $\diamondsuit$, and $\spadesuit$. This is not complete because the card could also be a heart.

When the outcomes are defined so that they are mutually exclusive and complete, the sum of the probabilities must equal 1. Also, any probability must be greater than or equal to 0 and less than or equal to 1. If an event has n outcomes denoted as O_1, O_2, and so on up to O_n, then

$$\mathrm{P}(O_1) + \mathrm{P}(O_2) + \cdots + \mathrm{P}(O_n) = 1$$

and each $\mathrm{P}(O_i) \geq 0$, $i = 1, 2, \ldots, n$.

E.1.5 Types of Probabilities

We will speak of three types of probability: ***marginal***, ***joint***, and ***conditional***. A ***marginal probability*** is simply what we have referred to as "probability" up to now. The adjective "marginal" comes from placing probabilities in a tabular form, in which some of the probabilities are placed in the margins around the table.

A ***joint probability*** refers to the probability of two outcomes occurring simultaneously. If the two outcomes are from the same event, the joint probability must be 0, because such outcomes must be mutually exclusive. The normal context, however, is when we are dealing with compound outcomes which have been grouped differently. For example, suppose that a card is dealt. One way to list the outcomes is by suit: $\clubsuit$, $\diamondsuit$, $\heartsuit$, and $\spadesuit$. Another way is to list the outcomes by rank: Ace, 2, 3, ..., Queen, and King. Yet another way is to list the outcomes by colour: red, and black. We will consider these to be three events: a suit event, a rank event, and a colour event.

If we pick any outcome from the suit event, and any outcome from the rank event, we see that the joint probability is the product of the two marginal probabilities. For example,

$$\begin{aligned}
\mathrm{P}(\spadesuit) &= \frac{13}{52} \\
&= \frac{1}{4} \\
\\
\mathrm{P}(\text{Jack}) &= \frac{4}{52}
\end{aligned}$$

$$\begin{aligned} &= \frac{1}{13} \\ \text{P(Jack}\spadesuit) &= \frac{1}{52} \end{aligned}$$

We see that

$$\text{P}(\spadesuit)\text{P(Jack)} = \left(\frac{1}{4}\right)\left(\frac{1}{13}\right) = \frac{1}{52} = \text{P(Jack}\spadesuit)$$

In situations like this, when the joint probability equals the product of the two marginal probabilities, we say that the two outcomes are ***independent***. In general, outcome i of one event and outcome j of another event are independent if and only if

$$\text{P}(O_i \& O_j) = \text{P}(O_i)\text{P}(O_j)$$

Things are quite different, however, if we look at the suit and colour events.

$$\begin{aligned} \text{P}(\spadesuit) &= \frac{1}{4} \\ &= 0.25 \\ \text{P(black)} &= \frac{26}{52} \\ &= 0.5 \end{aligned}$$

$$\begin{aligned} \text{P}(\spadesuit\text{\&black}) &= \frac{13}{52} \\ &= 0.25 \end{aligned}$$

These two outcomes are clearly not independent.

A ***conditional probability*** is written in the form P(Y/X), read as "the probability of Y given X". It states the probability of outcome Y occurring given that outcome X occurs. When X and Y are independent, P(Y/X) is simply P(Y). For example,

$$\text{P}(\spadesuit/\text{Jack}) = \frac{1}{4} = 0.25$$

which is the same as

$$\text{P}(\spadesuit) = \frac{13}{52} = 0.25$$

When the two outcomes are not independent, we cannot use the marginal probability alone. For example,

$$\text{P}(\spadesuit/\text{black}) = \frac{13}{26}$$

$$= \quad 0.5$$

Note that in general, P(Y/X) $\neq$ P(X/Y). Since all spades are black,

$$\begin{aligned} \text{P(black/}\spadesuit) &= \frac{13}{13} \\ &= 1 \\ &\neq \text{P(}\spadesuit\text{/black)} \end{aligned}$$

It is when outcomes are *not* independent that information about one outcome helps us to better determine what the other outcome is. For example, suppose that a card has been drawn, and someone is asked to guess whether or not it's a ♠. That person has a 0.25 chance of guessing correctly. If he or she is told "Here's a hint – it's a Jack", then because of independence, the "hint" doesn't help at all. He or she still has only a 25% chance of guessing correctly. However, he or she is told "Here's a hint – the card is black", then the probability of guessing correctly rises to 0.5.

The three types of probabilities, marginal, joint, and conditional are related as follows. The joint probability of X and Y is

$$\mathrm{P}(X\&Y) = \mathrm{P}(Y/X) \times \mathrm{P}(X) \tag{E.1}$$

It can also been found as

$$\mathrm{P}(X\&Y) = \mathrm{P}(X/Y) \times \mathrm{P}(Y) \tag{E.2}$$

Example

A university has recently admitted 500 students into the first year of four professional programs. The numbers by gender and school are

	Business	Engineering	Medicine	Nursing	Total
Male	95	110	25	10	240
Female	105	40	35	80	260
Total	200	150	60	90	500

A student's file is picked at random from among the 500. What is the probability that this file belongs to a male/female student? This is an event with two outcomes, male and female. The probabilities are found as the ratio of the relevant number of students.

$$\begin{aligned} \text{P(male)} &= \frac{\text{240 male students}}{\text{500 students in total}} \\ &= 0.48 \end{aligned}$$

$$\begin{aligned} \text{P(female)} &= \frac{260}{500} \\ &= 0.52 \end{aligned}$$

Note that

$$\text{P(male) + P(female)} = 0.48 + 0.52 = 1$$

Similarly, if we wish to know the probability that the student is majoring in business, this is

$$\begin{aligned} \text{P(business)} &= \frac{200}{500} \\ &= 0.4 \end{aligned}$$

The data can also be used to find joint probabilities. The probability that a student is both female and majoring in medicine is

$$\begin{aligned} \text{P(female \& medicine)} &= \frac{35}{500} \\ &= 0.07 \end{aligned}$$

Indeed, by dividing each of the original numbers by 500 (the total), we obtain a table giving the marginal and joint probabilities.

	Business	Engineering	Medicine	Nursing	Total
Male	0.19	0.22	0.05	0.02	0.48
Female	0.21	0.08	0.07	0.16	0.52
Total	0.40	0.30	0.12	0.18	1.00

The joint probabilities appear in the main body, while (as mentioned before) the marginal probabilities appear in the margins.

We can also find the conditional probabilities, either from the original data, or from the table of marginal and joint probabilities. For example, using the original data, the probability that a student is studying medicine, given that the student is female, is:

$$\begin{aligned} \text{P(medicine/female)} &= \frac{35}{260} \\ &= 0.1346 \end{aligned}$$

On the other hand, the probability that a student is female, given that the student is studying medicine, is:

$$\begin{aligned} \text{P(female/medicine)} &= \frac{35}{60} \\ &= 0.5833 \end{aligned}$$

Using the table of marginal and joint probabilities we have

$$\begin{aligned} \text{P(medicine/female)} &= \frac{0.07}{0.52} \\ &= 0.1346 \end{aligned}$$

$$\begin{aligned} \text{P(female/medicine)} &= \frac{0.07}{0.12} \\ &= 0.5833 \end{aligned}$$

E.1.6 Sequential Events

Up till now, we have only considered single events. Now we consider the situation where several events occur sequentially. We will look at the following: (1) tossing a coin three times, (2) rolling a die three times, (3) dealing three cards (without replacement) from a deck of cards, and (4) pulling socks (unseen) from a drawer until a pair of the same colour is obtained.

To analyze these situations, we will make use of what are called ***probability trees***. In a probability tree, each event is represented by a circle, and each outcome is represented by a branch. By convention, the tree grows from left to right. We will be placing joint probabilities inside the circles, so they are sized appropriately. In all cases, a probability of 1 will be written in the circle on the extreme left, meaning that there is a 100% chance that something will happen. Alongside each branch we place a verbal description of the outcome, and we give its probability. If the events are independent, then this will be a marginal probability. If the events are not independent, then this probability will be conditional on what has happened up to that point.

Except for the extreme left circle, the joint probability placed in any circle is the product of the probability on the branch to the left of that circle multiplied by the joint probability in the circle at the left end of the branch. Equivalently, at any circle, the joint probability is the product of all probabilities on the branches from the extreme left to that circle.

Tossing a Coin Three Times

When a coin is tossed three times, we have three independent events, each of which has two outcomes, heads and tails, of equal probability:
P(heads) = 0.5, and P(tails) = 0.5. We obtain the picture shown in Figure E.1.

Because of the symmetry of the probabilities, all eight final joint probabilities are the same, 0.125. If the order in which heads or tails are obtained does not matter, then we can combine some of the outcomes as follows: 3 heads; 2 heads, 1 tails (any order); 1 heads, 2 tails (any order); and 3 tails. Their probabilities are as follows:

3 heads	0.125
2 heads, 1 tails	0.375
1 heads, 2 tails	0.375
3 tails	0.125
Total	1.000

Flipping a Die Three Times

This is similar to the tossing of a coin, except that the probabilities are different. Suppose we are interested in obtaining a ‘two’ on the die, with all other numbers being lumped together as ‘Other’. The probability of obtaining a ‘two’ on any flip is $\frac{1}{6}$, and there is probability $\frac{5}{6}$ of obtaining ‘Other’. Since the fractions are not easy to express in decimal form, we will leave them as fractions. The probability tree for this situation is shown in Figure E.2.

If the order in which a ‘two’ or ‘Other’ is obtained does not matter, then we can combine some of the outcomes as follows: 3 twos; 2 twos, 1 other (any order); 1 two, 2 other (any order); and 3 other. Their probabilities are found by summing the appropriate final circles on the right of the diagram. as follows:

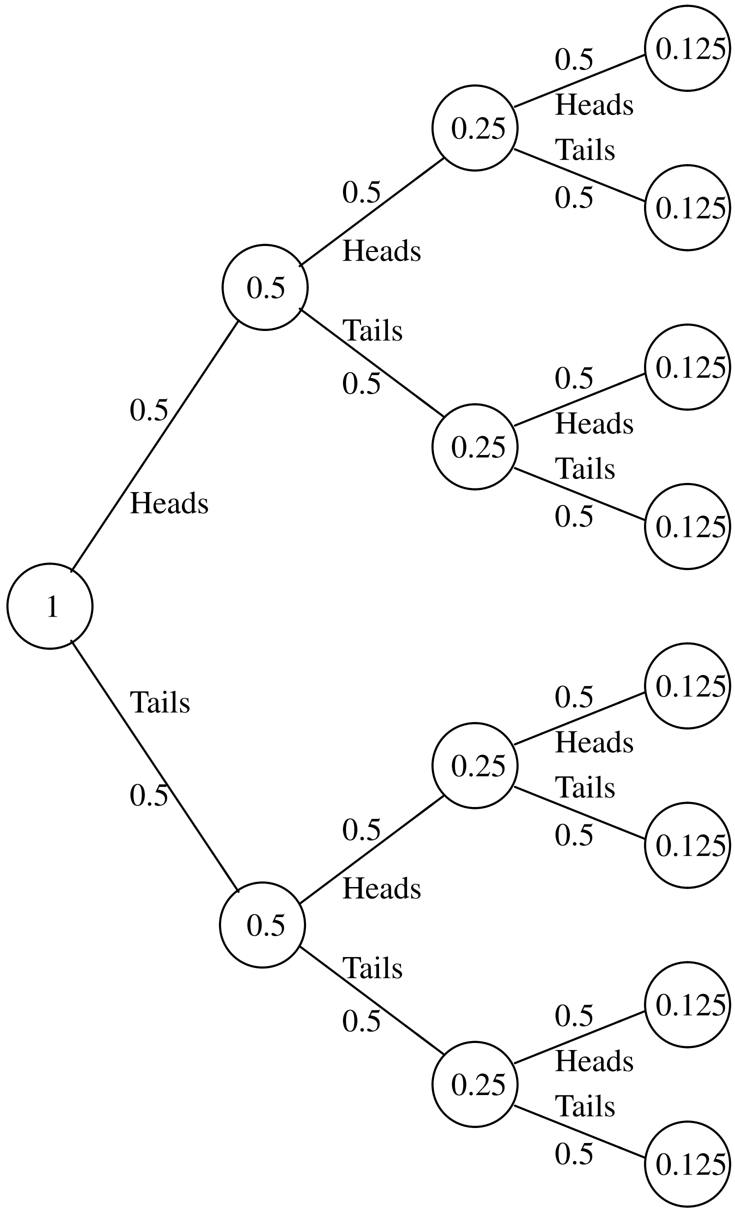


Figure E.1: Tossing a Coin Three Times

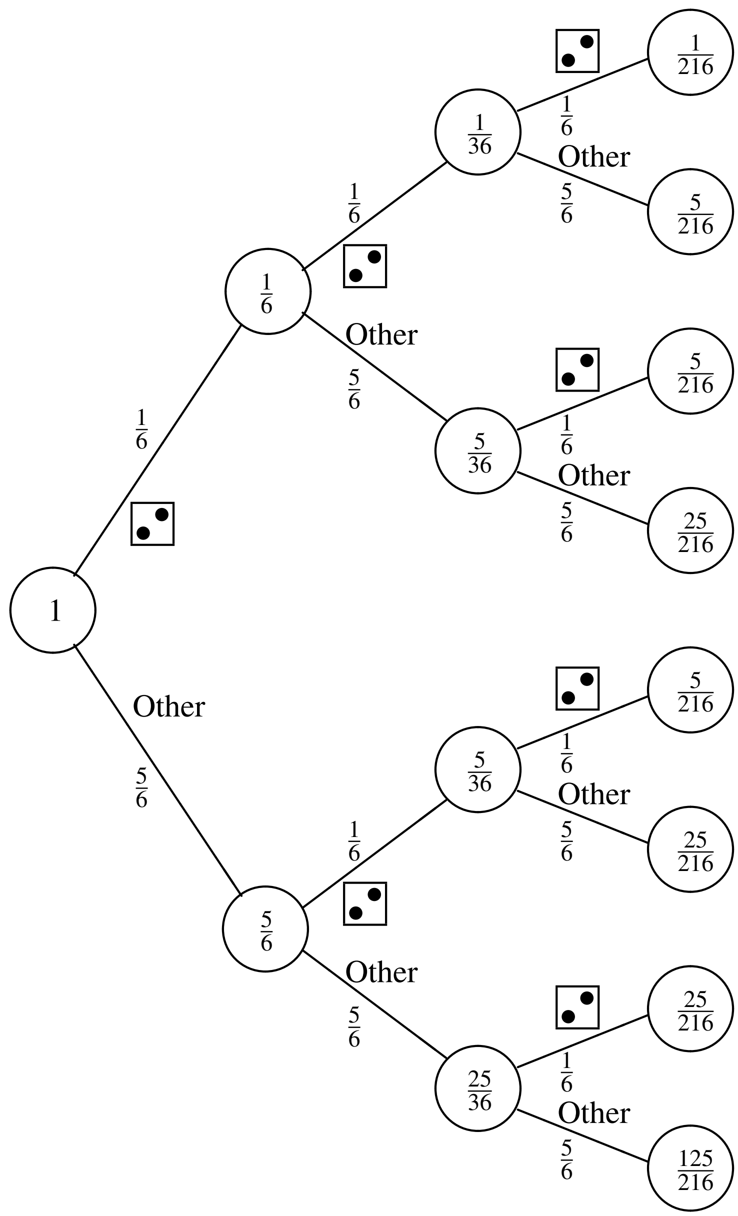


Figure E.2: Flipping a Die Three Times

3 twos		$\frac{1}{216}$
2 twos, 1 other	$3 \times \frac{5}{216} =$	$\frac{15}{216}$
1 two, 2 other	$3 \times \frac{25}{216} =$	$\frac{75}{216}$
3 other		$\frac{125}{216}$
Total	$\frac{216}{216} =$	1

Dealing Three Cards Without Replacement

The two preceding examples involved independent outcomes. For example, the probability of obtaining ‘heads’ on the third toss does not depend on what happened on the first two tosses of the coin. However, when dealing cards without replacement,[1] the outcomes are dependent. For example, if an Ace is dealt at the outset, then the probability that the second card dealt is an Ace is $\frac{3}{51}$, since three Aces remain in the deck which now has only 51 cards.

In the example which we consider here, we are looking at face cards (Kings, Queens, Jacks). We are dealing three cards without replacement, and will use a probability tree to obtain the probabilities of obtaining 0, 1, 2, or 3 face cards.

We begin with a standard deck of 52 cards, which contains 12 face cards (3 in each of the 4 suits). On the first deal, there is probability $\frac{12}{52}$ of obtaining a face card, and probability $\frac{40}{52}$ of obtaining any other card. On the second deal, there are 51 cards left in the deck. If the first card dealt was a face card, then there is probability $\frac{11}{51}$ of obtaining a face card now; however, if the first card dealt was not a face card, then there is probability $\frac{12}{51}$ of obtaining a face card now. The probability tree for this situation is shown in Figure E.3.

The probability of obtaining 3 face cards is $\frac{1320}{132{,}600}$ or 0.0099548. There are three ways to obtain exactly 2 face cards, each with probability $\frac{5{,}280}{132{,}600}$, hence the probability of obtaining 2 face cards is $\frac{3(5{,}280)}{132{,}600}$ or 0.1194570. There are three ways to obtain exactly 1 face card, each with probability $\frac{18{,}720}{132{,}600}$, hence the probability of obtaining 1 face card is $\frac{3(18{,}720)}{132{,}600}$ or 0.4235294. Finally, the probability of obtaining no face card is $\frac{59{,}280}{132{,}600}$ or 0.4470588. In summary:

[1]This is the normal way in which cards are dealt. Dealing *with* replacement would mean that a card is dealt, it is put back into the deck, the deck is re-shuffled, and then a card is dealt, and so on. When dealing with replacement, it is possible to obtain the same card twice.

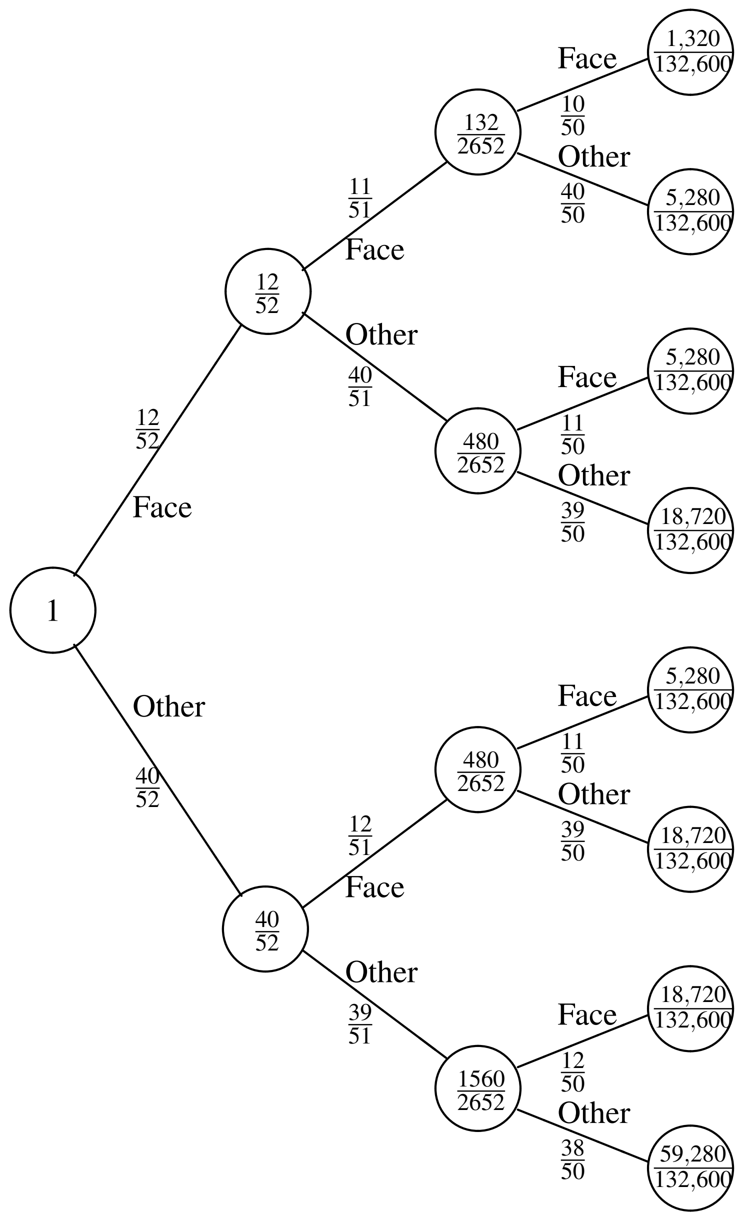


Figure E.3: Dealing Three Cards Without Replacement

Number of Face Cards	Probability
3	0.0099548
2	0.1194570
1	0.4235294
0	0.4470588
Total	1.0000000

From this table other values of interest may be obtained. For example, the probability of obtaining at least two face cards is $P(2)+P(3)$, which is

$$0.1194570+0.0099548=0.1294118$$

The probability of obtaining at least one face card can be found directly as $P(1)+P(2)+P(3)$, or it may be found indirectly as $1-P(0)$:

$$1-0.4470588=0.5529412$$

Socks of the Same Colour Problem

Problem Description

A small girl reaches into the top drawer of a chest of drawers to obtain a pair of socks. The drawer contains two pairs of blue socks (i.e. four socks) and three pairs of red socks. The socks have not been folded into matched pairs, but instead the ten socks lie randomly scattered about the drawer. She wishes to obtain two socks of the same colour, but she is too short to be able to see the socks in the drawer. She pulls out two socks. If both are of the same colour, she wears them. Otherwise, she pulls out a third sock, which will, of course, guarantee that she will obtain a pair of socks of the same colour.

What is the probability that she will obtain a blue pair of socks by this process?

Solution

While there may be only one physical movement which pulls the first two socks from the drawer, it is useful to think of two socks being pulled sequentially, since this aids in the calculation of the probabilities. It is very useful in a problem such as this to draw a probability tree. Unlike our previous trees, there may be two or three branches from start to finish. Like the card problem, we do not have independence: at the outset, there are 4 chances in 10 of pulling a blue sock, but if the first sock is blue, then the probability of obtaining a blue sock on the second pull falls to 3 chances in 9, but if the first sock was red, then the chance of obtaining a blue sock on the second pull rises to 4 in 9.

The probability tree for the problem is shown in Figure E.4.

There are three ways to obtain a blue pair of socks; the joint probabilities of these three ways are found on the probability tree.

1. two blue socks in a row, with probability $\frac{12}{90}$
2. blue, then red, then blue, with probability $\frac{72}{720}$
3. red, then two blue socks in a row, with probability $\frac{72}{720}$

The probability of obtaining a blue pair of socks by any means is the sum of these three probabilities.

$$\begin{aligned}
\text{P(blue pair)} &= \frac{12}{90}+\frac{72}{720}+\frac{72}{720} \\
&= \frac{12}{90}+\frac{9}{90}+\frac{9}{90} \\
&= \frac{30}{90} \\
&= \frac{1}{3} \text{ or } 0.333...
\end{aligned}$$

As this example illustrates, trees are very useful for structuring problems. However, there are faster ways of solving problems when a common pattern emerges, such as in the coin and die examples. We shall see in the next chapter that when this happens, a formula may be derived as an alternative to a tree.

E.1.7 Summary

Such things as the flip of a coin, the toss of a die, or the dealing of a card are called events, for which there are 2, 6, and 52 possible outcomes respectively. For such events, we make the Laplacian assumption that each outcome is equally likely. For most events, however, the Laplacian assumption is not appropriate. For example, the outcomes “oil is found” and “oil is not found” are not equally likely. In a case such as that, the probability of finding oil is a subjective estimate.

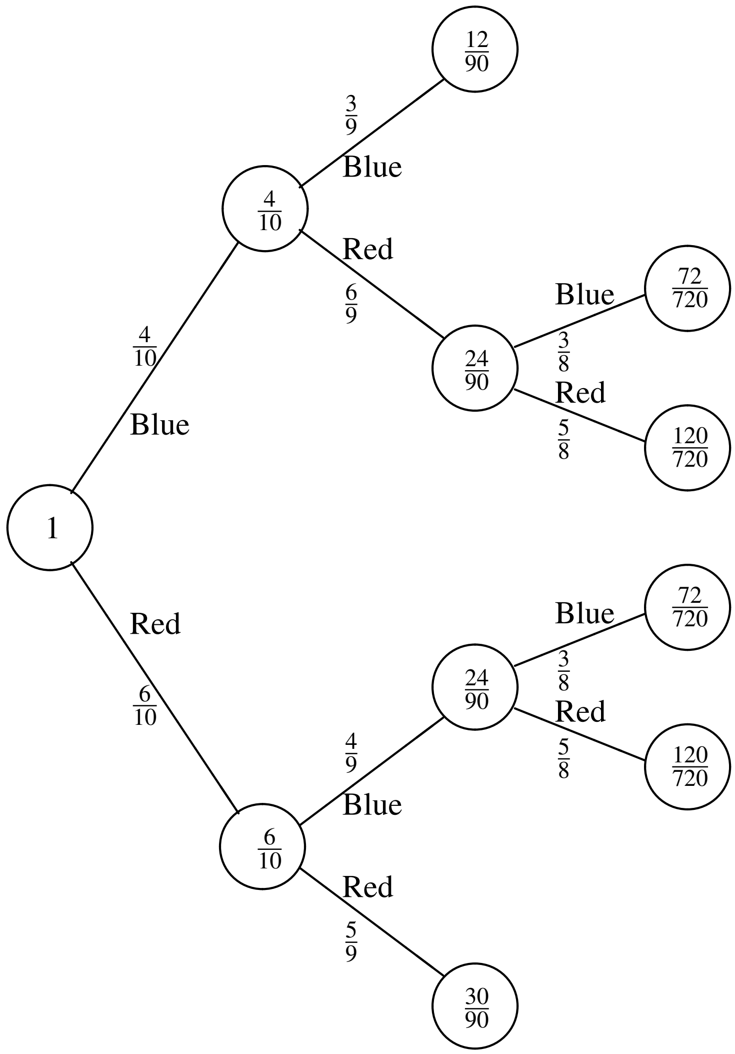


Figure E.4: Socks of the same Colour Problem

Even with events such as the deal of a card, we may wish to group the outcomes; when this happens they are usually no longer Laplacian. For example, when a card is dealt it is either a ♣, or it is not. The probabilities of these outcomes are obtained by counting the number of ways each could occur divided by the number of ungrouped outcomes, e.g.

$$P(\clubsuit) = \frac{13}{52} = 0.25$$

Every probability is a number between 0 and 1 inclusive. Furthermore, the sum of the probabiliites of the outcomes of an event must be 1.

We considered three types of probabilities: marginal, conditional, and joint. When the outcomes are independent, the joint probability is the product of the two marginal probabilities. When independence does not hold, the joint probability is the product of conditional and marginal probabilities.

When we have two or more sequential events, a probability tree can be drawn to help describe the situation, the sequence being shown from left to right. A circle is used as a symbol for an event, and lines (branches) coming out of the circle represent the outcomes of the event. Words can be placed on the tree to indicate what is being described. Marginal and conditional probabilities are written on the branches of the tree, and joint probabilities are computed and written in the circles. One joint probability or the sum of several joint probabilities on the right-hand branches will represent something from all the events, such as the number of Queens obtained in a deal of five cards. Because of the large amount of work required to use them, probability trees are most useful for one-of-a-kind situations.

E.1.8 Problems for Student Completion

1. What error exists in each of the following statements?

   (a) There's a 37% chance of high sales, a 44% chance of medium sales, and a 29% chance of low sales.

   (b) When five cards are dealt, the number of red cards received must be between one and five inclusive.

   (c) A spacecraft begins its journey with two radios operating. There is a 1% chance that either radio will fail; the failure of one radio is independent of the failure of the other. Hence there is a 98% chance that both radios will remain working.

2. A restaurant chain has the following promotion. The customer is given a card with five rectangular areas (‘boxes’). In order to see what a box says, it must first be scratched. Three boxes say “you lose”, one says “you win”, and the other one says “scratch again”. The customer scratches boxes until either “you lose” or “you win” appears [i.e. the card is void if both “you win” and “you lose” have been scratched].

   (a) What is the probability that the customer would win?

   (b) If the card had two “you lose” and two “scratch again” boxes (still one “you win”), what is the probability that the customer would win?

3. Professor John Smith teaches two undergraduate statistics courses. The class of statistics 2500 has forty second year students and ten third year students. The more advanced statistics 2501 has twenty second year students and thirty third year students.

   As an example of a business sampling technique, Prof. Smith randomly selects a name from the class list for statistics 2500. If the name is that of a student who is in his or her second year, then Prof. Smith will again select a name from the statistics 2500 list (possibly the same one as before); otherwise, he will randomly select a name from the statistics 2501 list.

   What is the probability that:

   (a) both students are in their second year?

   (b) both students are in their third year?

   (c) one student is in his or her second year and the other is in his or her third year, regardless of order?

4. Repeat the above problem, except that now the first name selected is scratched off the list of statistics 2500 students, and hence it cannot be selected twice.

5. A journey by plane from a city to a remote area, or from the remote area back to the city, requires 8 hours flying time and can only be done in fair weather. Because of a concern for crew fatigue, the plane cannot make a same-day return flight. The pilot will fly out on the first fair day, and will return on the next fair day after that. Over the next three days, the weather forecast gives a probability for fair weather of 20% tomorrow, 60% for the second day, and 35% for the third day.

(a) Draw a probability tree for this situation, ending a branch if they return to the city, or if three days have elapsed, whichever comes first.

(b) What is the probability that over the next three days

(i) the plane will never leave the city?
(ii) the plane makes it to the remote area, but not home again?
(iii) the plane is able to fly out to the remote area and return to the city?

6. Repeat the previous problem with the following information. On the first day, the probability of fair weather is, as before, 20%. However, on the second day, the probability of fair weather is only 10% if day 1's weather was foul, but is 70% if day 1's weather was fair. On the third day, there is a 95% chance of fair weather if both previous days were fair, there is a 55% chance of fair weather is either previous day was fair, and there is a 99% chance of foul weather if both previous days were foul.

E.1.9 Answers

1. Hints: (a) What is the sum of the probabilities? (b) What happens if there are five black cards? (c) Draw a probability tree. Find the joint probability that neither radio will fail.

2. (a) 0.25 (b) $\frac{1}{3}$

3. (a) 0.64 (b) 0.12 (c) 0.24

4. (a) 0.6367 (b) 0.12 (c) 0.2433

5. (b) (i) 0.208 (ii) 0.476 (iii) 0.316

6. (b) (i) 0.7128 (ii) 0.0702 (iii) 0.217

E.2 Sensitivity Analysis (Payoff Matrices)

The subject of *sensitivity analysis* (also called *what-if analysis*) is a recurring theme throughout the field of management science. The whole point to building a model is that it is much cheaper than building what the model represents. We can play around with the model quite inexpensively, and one of the things that we should do is see how sensitive it is to changes in the built-in assumptions of the model.

The Greek symbol Δ (pronounced *delta*) is often used to represent a change to something. We might use Δp (read as "delta p") to represent a change in probability,[2] or Δc to represent a change in cost; where the context is clear, we can simply use Δ. Usually, Δ can be either positive or negative. To keeps things simple, we often just vary one parameter at a time, but changing probabilities is an important exception. If one probability is increased, then at least one other probability must be decreased (by the same absolute amount) so that the probabilities remain summed to one. Also, in this situation, we must establish a domain for Δ based on the fact that no probability can go below 0 or above 1.

E.2.1 Theatre Example

Included in the original parameters are that the probability of fringe interest is 0.2, and the probability of average interest is 0.7. Suppose that we now wish to see what happens if we vary these two probabilities, with everything else remaining constant. Suppose that the first is increased by Δ, and the other is decreased by Δ. (Doing it the other way around would be fine; everything will work out in the end.) Hence we have:

$$\begin{aligned} p(\text{fringe}) &= 0.2 + \Delta \\ p(\text{average}) &= 0.7 - \Delta \end{aligned}$$

We must ensure that neither probability goes below 0. If we do this, we will automatically ensure that neither probability goes above 1. The condition that $0.2 + \Delta \geq 0$ will be true provided that $\Delta \geq -0.2$. The condition that $0.7 - \Delta \geq 0$ will be true provided that $\Delta \leq 0.7$. Hence, the domain of Δ is:

$$-0.2 \leq \Delta \leq 0.7$$

[2] Note that in this context Δp is simply one construct; it does not mean Δ multiplied by p.

[Note: In general if the two probabilities are $a+\Delta$ and $b-\Delta$, then we must have $-a \leq \Delta \leq b$.]

When we first solved the problem we obtained:

			Demand for Tickets				
Theatre	3-Night		Fringe	Average	Great	Heavy	Expected
Size	Capacity	Rent	250	800	2300	4500	Value
Small	300	\$600	1900	2400	2400	2400	\$2300
Medium	1200	\$1800	700	6200	10,200	10,200	\$5500
Large	3600	\$4700	−2200	3300	18,300	31,300	\$3830
		Probability	0.20	0.70	0.09	0.01	

Now, in the probability row, the 0.20 becomes $0.20+\Delta$, and the 0.70 becomes $0.70-\Delta$, and we wish to determine the revised Expected Values.

			Demand for Tickets				
Theatre	3-Night		Fringe	Average	Great	Heavy	Expected
Size	Capacity	Rent	250	800	2300	4500	Value
Small	300	\$600	1900	2400	2400	2400	
Medium	1200	\$1800	700	6200	10,200	10,200	
Large	3600	\$4700	−2200	3300	18,300	31,300	
		Probability	0.20	0.70	0.09	0.01	
			$+\Delta$	$-\Delta$			

The long way to do this, using the "Small" alternative to illustrate, is to re-compute the entire dot product ("Small" row and the Probability row).

$$\begin{aligned}
\text{EV(small)} &= (0.2+\Delta)1900+(0.7-\Delta)2400+.09(2400)+.01(2400)\\
&= 0.2(1900)+1900\Delta+0.7(2400)-2400\Delta+.09(2400)+.01(2400)\\
&= 0.2(1900)+(0.7+0.09+0.01)2400+(1900-2400)\Delta\\
&= 2300-500\Delta
\end{aligned}$$

The short way to do this is to recognize that the "2300" has been computed already – all we need to do is include the terms involving "Δ". [If $\Delta=0$, we must obtain the original result.] All we need are the columns which contain the Δ's and the original Expected Values.

	Fringe	Average	Expected Value Original	Expected Value Δ Term
Small	1900	2400	2300	
Medium	700	6200	5500	
Large	-2200	3300	3830	
	Δ	$-\Delta$		

The short way is simply:

$$\begin{aligned}
\text{EV(small)} &= 2300 + 1900\Delta + 2400(-\Delta) \\
&= 2300 + 1900\Delta - 2400\Delta \\
&= 2300 - 500\Delta
\end{aligned}$$

For the medium and large theatre alternatives we have:

$$\begin{aligned}
\text{EV(medium)} &= 5500 + 700\Delta + 6200(-\Delta) \\
&= 5500 - 5500\Delta
\end{aligned}$$

$$\begin{aligned}
\text{EV(large)} &= 3830 + (-2200)\Delta + 3300(-\Delta) \\
&= 3830 - 5500\Delta
\end{aligned}$$

The completed table is:

	Fringe	Average	Expected Value Original	Expected Value Δ Term
Small	1900	2400	2300	-500Δ
Medium	700	6200	5500	-5500Δ
Large	-2200	3300	3830	-5500Δ
	Δ	$-\Delta$		

Comparing Medium with Large, we see that for any value of Δ,

$$5500 - 5500\Delta > 3830 - 5500\Delta$$

and hence Medium is better than Large. These lines are parallel (because of the -5500) and therefore they never intercept.

If we compare Small with Medium, we have EV(Small) = $2300 - 500\Delta$ versus EV(Medium) = $5500 - 5500\Delta$. We are *indifferent* between two alternatives when neither is preferred to the other. To find the point of indifference, we set the two payoffs equal to each other:

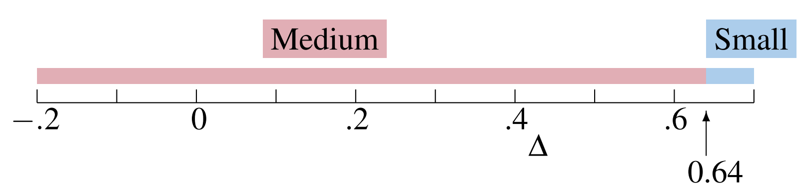
$$\begin{aligned} \text{EV(Small)} &= \text{EV(Medium)} \\ 2300 - 500\Delta &= 5500 - 5500\Delta \\ 5000\Delta &= 3200 \\ \Delta &= 0.64 \end{aligned}$$

This value is within the domain $-0.2 \leq \Delta \leq 0.7$. [Were it not so, there would be no point of indifference.]

We know that Medium is preferred at $\Delta = 0$ (the current situation), and we have found that we would switch to Small at $\Delta = 0.64$. Since these are the only alternatives (because Large was eliminated), Medium must be best for any Δ in the domain < 0.64, there's a tie at 0.64, and Small is best for all other values. By letting both alternatives be considered "best" at the tie, we can state the regions of preference as:

$$\begin{array}{rr} -0.2 \leq \Delta \leq 0.64 & \text{Medium} \\ 0.64 \leq \Delta \leq 0.70 & \text{Small} \end{array}$$

We can also show this information on a number line for Δ (where $-0.2 \leq \Delta \leq 0.7$), highlighting with colour the regions for the recommended theatre size.



$\Delta = 0.64$ is a very large change for a probability. If we believe that the initial estimate of 0.2 couldn't be off the true value by all that much, then we would be quite confident that our initial choice of Medium is correct.

The point of indifference can also be expressed in terms of the original probabilities. These are:

$$p(\text{fringe}) = 0.2 + 0.64$$

$$= 0.84$$

$$\begin{aligned} p(\text{average}) &= 0.7 - 0.64 \\ &= 0.06 \end{aligned}$$

E.2.2 A More Complicated Example

The preceding example was fairly easy in that we were able to reduce it down to two alternatives, and so we only had to find a single point of indifference. Usually, however, we cannot eliminate any alternative simply by inspection. When this happens, a conceptually easy approach is to make a graph of EV versus Δ. Doing it first this way gives us a shorter analytical method for this type of problem. The example presented here provides an illustration of these concepts.

Consider an example with four alternatives and three outcomes, for which we begin with all payoffs having been found, and the expected values having been calculated:

	O_1	O_2	O_3	EV
A_1	7	5	4	5.1
A_2	5	5	6	5.3
A_3	4	6	3	4.7
A_4	6	4	6	5.0
Prob.	.2	.5	.3	

Hence the recommendation is to choose alternative A_2, with an expected payoff of 5.3. Now suppose that we wish to vary the probabilities for O_2 and O_3. We will let the probability of O_2 be $0.5+\Delta$, and the probability of O_3 be $0.3-\Delta$. The domain for Δ is therefore:

$$-0.5 \leq \Delta \leq 0.3$$

The new expected values are:

	O_2	O_3	EV
A_1	5	4	$5.1+\Delta$
A_2	5	6	$5.3-\Delta$
A_3	6	3	$4.7+3\Delta$
A_4	4	6	$5.0-2\Delta$
	Δ	$-\Delta$	

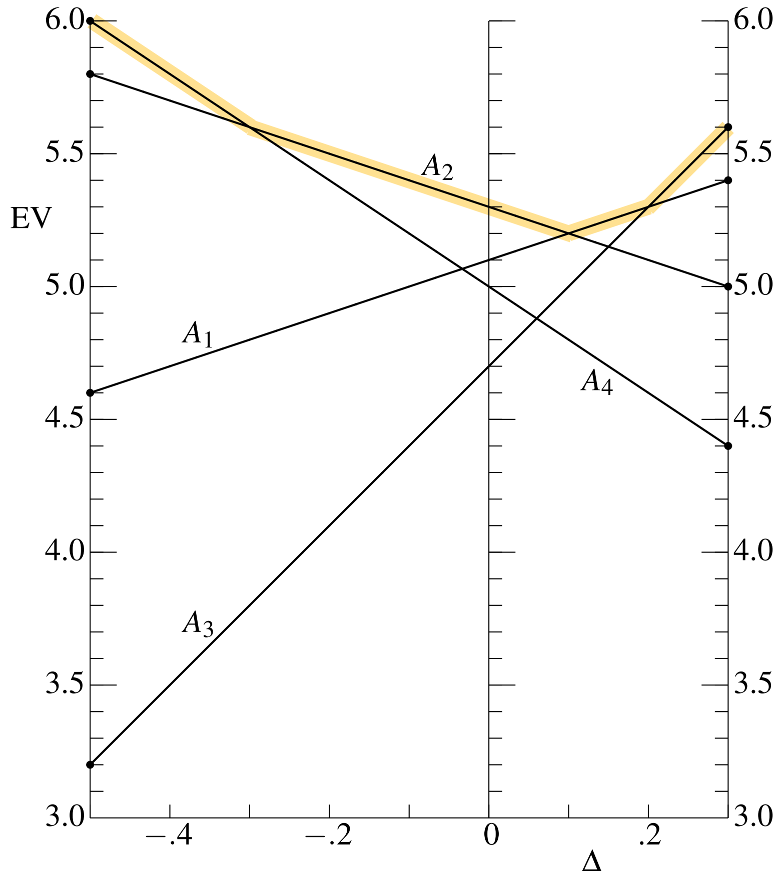
Unlike the previous example, it is difficult to remove an alternative simply by inspection; one approach is to draw a graph.

Graphical Solution Each of the EV equations is a straight line. For each of these, all we need do is find the EV for any $\Delta \neq 0$, and this along with the original EV (i.e. at $\Delta = 0$) gives the two distinct points needed to define the line. However, with a little bit of extra work we can obtain each line with a check on the calculations. To do this we find the EV at the lower limit for Δ, and then find the EV at the upper limit for Δ. These two points are used to define the line, and the check on the calculations comes from making sure that the line passes through the original EV at $\Delta = 0$. In this example, the lower and upper limits for Δ are at -0.5 and 0.3 respectively. For alternative 1, the EV ranges from $5.1 + (-0.5) = 4.6$ to $5.1 + 0.3 = 5.4$. For alternative 2, the EV ranges from $5.3 - (-0.5) = 5.8$ to $5.3 - 0.3 = 5.0$. For alternative 3, the EV ranges from $4.7 + 3(-0.5) = 3.2$ to $4.7 + 3(0.3) = 5.6$. Finally, for alternative 4 the EV ranges from $5 - 2(-0.5) = 6.0$ to $5 - 2(0.3) = 4.4$. In summary we have:

	EV at		
	$\Delta = -0.5$	$\Delta = 0$	$\Delta = 0.3$
A_1	4.6	5.1	5.4
A_2	5.8	5.3	5.0
A_3	3.2	4.7	5.6
A_4	6.0	5.0	4.4

The horizontal axis goes from $\Delta = -0.5$ to $\Delta = 0.3$. Since the smallest EV is 3.2, and the largest is 6.0, we can save vertical space by having the axis run only between 3 and 6 (rather than starting at 0). It is helpful to draw this graph with three vertical axes: one through the lower limit for Δ; one through 0; and one through the upper limit for Δ.

With the axes drawn, we proceed with drawing the four lines. The A_1 line goes from 4.6 on the left vertical axis to 5.4 on the right vertical axis. Next to this line the A_1 symbol is drawn. On the centre vertical axis, we see indeed that the line passes through the point (0, 5.1). The other three lines are drawn with their symbols, each time verifying that the point on the centre vertical axis is where it should be.



We want, of course, the line segments which maximize the expected value. These line segments have been highlighted on the graph. At $\Delta = -0.5$, A_4 is best. As we move to the right, the best alternative switches to A_2, then A_1, and then A_3. Hence we need to find where the following pairs of lines intercept: A_4 and A_2; A_2 and A_1; and A_1 and A_3.

To find the value of Δ at which the A_4 and A_2 lines intercept, we set the EV equations equal to each other:

$$\begin{aligned} \mathrm{EV}(A_4) &= \mathrm{EV}(A_2) \\ 5.0 - 2\Delta &= 5.3 - \Delta \\ -\Delta &= 0.3 \end{aligned}$$

$$\Delta = -0.3$$

At this value for Δ, $\mathrm{EV}(A_4) = 5.0 - 2(-0.3) = 5.6$. [Also, $\mathrm{EV}(A_2) = 5.3 - (-0.3) = 5.6$.] Hence the two lines intercept at $(-0.3,\ 5.6)$.

We then find the other two interception points:

$$\begin{aligned}
\mathrm{EV}(A_2) &= \mathrm{EV}(A_1) \\
5.3 - \Delta &= 5.1 + \Delta \\
-2\Delta &= -0.2 \\
\Delta &= 0.1
\end{aligned}$$

At this value for Δ, $\mathrm{EV}(A_2) = 5.3 - (0.1) = 5.2$.

$$\begin{aligned}
\mathrm{EV}(A_1) &= \mathrm{EV}(A_3) \\
5.1 + \Delta &= 4.7 + 3\Delta \\
-2\Delta &= -0.4 \\
\Delta &= 0.2
\end{aligned}$$

At this value for Δ, $\mathrm{EV}(A_1) = 5.1 + (0.2) = 5.3$.

There are three ways that we could report these values. As we did in the previous example, we could draw a number line, highlighting the regions where a particular alternative is best. Secondly, we could indicate this information on the graph, which of course gives the absolute rather than just the relative ranking of each alternative. Thirdly, we could simply report the regions as follows:

Region for Δ	Best Alternative
$-0.5 \leq \Delta \leq -0.3$	A_4
$-0.3 \leq \Delta \leq 0.1$	A_2
$0.1 \leq \Delta \leq 0.2$	A_1
$0.2 \leq \Delta \leq 0.3$	A_3

Sometimes, we do not need to know the best alternatives over the entire domain of Δ. Instead, we might only wish to determine the values for Δ for which the current solution (i.e. at $\Delta = 0$) remains optimal. For this example, the current solution remains A_2 provided that:

$$-0.3 \leq \Delta \leq 0.1$$

Analytical Solution We can also solve such a problem quickly as follows. First, here are some general comments for any situation. Consider two alternatives, with expected payoffs $a+b\Delta$, and $c+d\Delta$, where $a>c$ and $b \neq d$ (i.e. the two lines are not parallel). These alternatives have the same expected payoff when:

$$\begin{aligned} c+d\Delta &= a+b\Delta \\ (d-b)\Delta &= a-c \\ \Delta &= \frac{a-c}{d-b} \end{aligned}$$

If the critical value $(a-c)/(d-b)$ turns out to be outside of the domain of Δ, then the $c+d\Delta$ alternative is not best for any value of Δ. If, however, it is inside the domain, then we must consider this alternative along with any others.

With many alternatives we find the critical value of Δ for each one (where the comparison alternative is the one optimal at $\Delta = 0$); we seek the ones whose critical values are immediately on either side of 0.

Now we solve the example from before, first going back to the table showing the effects of Δ.

	O_2	O_3	EV
A_1	5	4	$5.1+\Delta$
A_2	5	6	$5.3-\Delta$
A_3	6	3	$4.7+3\Delta$
A_4	4	6	$5.0-2\Delta$
	Δ	$-\Delta$	

At $\Delta = 0$, A_2 is best. Hence we find where the A_1, A_3, and A_4 lines meet the A_2 line:

$$\begin{aligned} \text{EV}(A_2) &= \text{EV}(A_1) \\ 5.3-\Delta &= 5.1+\Delta \\ -2\Delta &= -0.2 \\ \Delta &= 0.1 \end{aligned}$$

$$\begin{aligned} \text{EV}(A_2) &= \text{EV}(A_3) \\ 5.3-\Delta &= 4.7+3\Delta \\ -4\Delta &= -0.6 \\ \Delta &= 0.15 \end{aligned}$$

$$\begin{aligned}
\mathrm{EV}(A_2) &= \mathrm{EV}(A_4) \\
5.3 - \Delta &= 5.0 - 2\Delta \\
\Delta &= -0.3
\end{aligned}$$

Comparing the critical values 0.1, 0.15, and -0.3, the ones immediately on either side of 0 are -0.3 (line A_4) and 0.1 (line A_1). Hence A_2 remains optimal from -0.3 to 0.1. Below -0.3, A_4 is best, and just above 0.1, A_1 is best. Further on, the A_1 and A_3 lines will intercept:

$$\begin{aligned}
\mathrm{EV}(A_1) &= \mathrm{EV}(A_3) \\
5.1 + \Delta &= 4.7 + 3\Delta \\
-2\Delta &= -0.4 \\
\Delta &= 0.2
\end{aligned}$$

Over the entire domain of Δ we have:

Region for Δ	Best Alternative
$-0.5 \le \Delta \le -0.3$	A_4
$-0.3 \le \Delta \le 0.1$	A_2
$0.1 \le \Delta \le 0.2$	A_1
$0.2 \le \Delta \le 0.3$	A_3

E.2.3 Sensitivity Problem 1

In this problem we use the data of the computer example (Problems for Student Completion, Problem 2 on page 387) with salvage value, except that we now allow the probabilities of demand for 10 and demand for 14 computers to vary. Make a graph of Expected Value vs. Δ for the five alternatives, and from this determine all the regions of Δ where one of the alternatives is better than the others.

E.2.4 Sensitivity Problem 2

	O_1	O_2	O_3	EV
A_1	8	2	4	
A_2	9	7	3	
A_3	70	15	-30	
A_4	-40	60	20	
Prob.	.3	.1	.6	

(a) Find the best alternative, using expected value as the decision criterion.

Now suppose that the probability of O_1 increases by Δ, the probability of O_2 increases by 3Δ, the probability of O_3 decreases by 4Δ.

(b) What is the domain of Δ?

(c) Find, by the analytical method, the regions of Δ where each alternative is best.

E.3 Sensitivity Analysis (Decision Trees)

Here we look at two types of sensitivity analysis applied to the New Detergent case which was analyzed at the outset of the chapter. While we could examine the effect of changing one or more of the parameters by any amount, usually we are only interested in finding the point(s) at which the recommendation would change. This is equivalent to finding the endpoints for which the proposed change does *not* alter the current recommendation. First, we will look at changing costs, and then we shall look at changing probabilities.

E.3.1 Changing Costs

The effect of changing the cost of making the ads is very easy to analyze. From Figure 9.4, we see that not making the ads leads to a payoff of 0, and the payoff at the square on the right of the make ads alternative branch has a ranking payoff of \$171,700. Therefore, the ads can cost up to \$171,700 before making the other alternative better. At the other extreme, if the ads cost nothing then the *make ads* alternative is of course still preferred. Since the ads currently cost \$15,000, we could say that the cost could be decreased by \$15,000 or increased by \$156,700 without affecting the current recommendation.

Alternatively we could use the concept of a change Δ. We can think of the cost of the ads as being $15+\Delta$, where Δ is the change to the cost in thousands of dollars, and where the context requires that Δ be ≥ -15. The payoff at the initial node is $156.7-\Delta$ (i.e. $171.7-(15+\Delta)$, and to keep the recommendation unchanged we require that $156.7-\Delta$ be at least as good as the payoff of the other alternative (which is 0), i.e. $156.7-\Delta \geq 0$, or $\Delta \leq 156.7$. In other words, the cost could be decreased by \$15,000 or increased by \$156,700 without affecting the current recommendation.

The effect of changing the cost of the test market campaigns is a bit trickier. While only one cost is being changed, two payoffs are affected by this change. Suppose that the cost (in thousands of dollars) of a test market campaign is now $10+\Delta$ (where Δ is in thousands of dollars and $\Delta \geq -10$). The cost next to the alternative branch for testing in two markets is therefore $20+2\Delta$. Therefore, the ranking payoff at the square labelled "Number of Test Markets" is either $181.70-(10+\Delta)$ or $188.2224-(20+2\Delta)$, *whichever is higher*. These expressions simplify to $171.70-\Delta$ and $168.2224-2\Delta$ respectively. We now need to find the values for Δ for which the currently better alternative remains better.

$$\begin{aligned} 171.70-\Delta &\geq 168.2224-2\Delta \\ \Delta &\geq -3.4776 \end{aligned}$$

In other words, as long as the cost of one test market campaign does not fall by more than \$3,477.60, we keep testing in just one test market. Conversely, if the cost falls by more than this amount (i.e. if the cost per test market becomes less than $\$10,000-\$3,477.60=\$6522.40$), then the recommended solution is to test in two markets.

We must also check what happens in the rollback. With the same recommendation (use one test market), we have $171.7-\Delta$ at the square labelled "Number of Test Markets". This is turn causes the payoff at the initial square to fall to $156.7-\Delta$, and the recommendation at this initial square stays the same for $\Delta \leq 156.7$.

In summary, if $\Delta > -3.4776$, then one test market is preferred to two, and if $\Delta \leq 156.7$, then the company prefers testing in one market to doing nothing. In other words the recommendation remains unchanged provided that:

$$-3.4776 \leq \Delta \leq 156.70$$

The current cost of the test market campaign is \$10,000. Hence, in absolute terms, the recommendation remains unchanged provided that the cost of the test market campaign remains between \$6522.40 and \$166,700.

E.3.2 Changing Probabilities

To illustrate the effect of changing probabilities, consider the probabilities of success and failure in the national campaign, after two markets have been tested, and where one success and one failure has been obtained. These numbers are currently

0.35 and 0.65 for success and failure respectively. Now we will let the probability of success be $0.35+\Delta$, and hence the probability of failure is $0.65-\Delta$. To prevent a probability from going below 0 or above 1, we must place the condition that $-0.35 \leq \Delta \leq 0.65$. These adjustments are shown on the appropriate outcome branches on the right-hand side of Figure E.5.

These changes cascade through the tree, affecting most of the ranking payoffs. At the two bottom circles on the right, we increase the payoffs by $4000\Delta + 400(-\Delta) = 3600\Delta$. The 1660 figure does not need to be recomputed – this is the advantage of dealing with changes to the current probabilities rather than looking at absolute probabilities. At the squares immediately to the left, the payoffs will also increase by 3600Δ, provided that the "proceed" alternative remains better than the "abandon" alternative. This will be the case provided that:

$$810+3600\Delta \geq 0$$

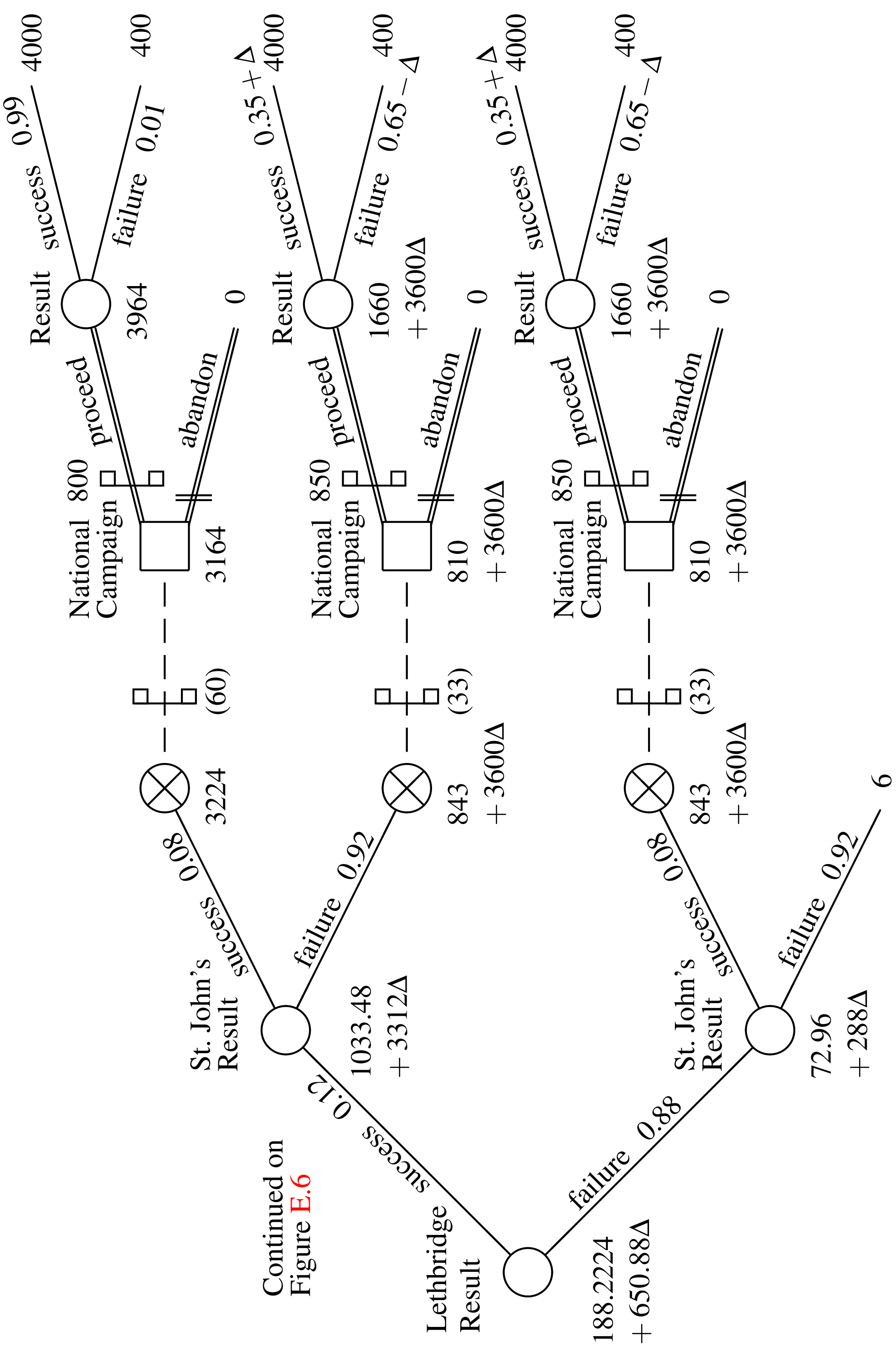


Figure E.5: Sensitivity Analysis – Second Continuation ($\Delta \geq -0.225$)

This condition simplifies to $\Delta \geq -0.225$. If Δ goes below this figure, then the abandon alternative would be preferred to the proceed alternative, but this change would not affect the overall recommendation, because this part of the tree is not part of the current recommendation. Now let us suppose that $\Delta \geq -0.225$, and see what affect this has on the rest of the tree. The bottom two payoff nodes also increase by 3600Δ, and then the rollback increases the payoffs by $0.92(3600\Delta) = 3312\Delta$ after a success in Lethbridge, and by $0.08(3600\Delta) = 288\Delta$ after a failure in Lethbridge. Finally, we obtain an increase of $0.12(3312\Delta) + 0.88(288\Delta) = 650.88\Delta$ at the circle on the extreme left, and this increase is transferred to the appropriate place on Figure E.6.

For the current recommendation to remain unchanged, we must have

$$\begin{aligned} 171.70 &\geq 188.2224 + 650.88\Delta - 20 \\ 171.70 &\geq 168.2224 + 650.88\Delta \\ 3.4776 &\geq 650.88\Delta \\ 0.00534... &\geq \Delta \end{aligned}$$

Hence the recommendation remains unchanged provided that $\Delta \leq 0.00534$. This is not much, when the current probability of success nationally (after one success, and one failure) is 0.35, with all probabilities being reported to the nearest 5%. All it would take is an increase to say 36%, and the recommendation would change to testing in two markets, and then proceeding if at least one of these turns out to be a success.

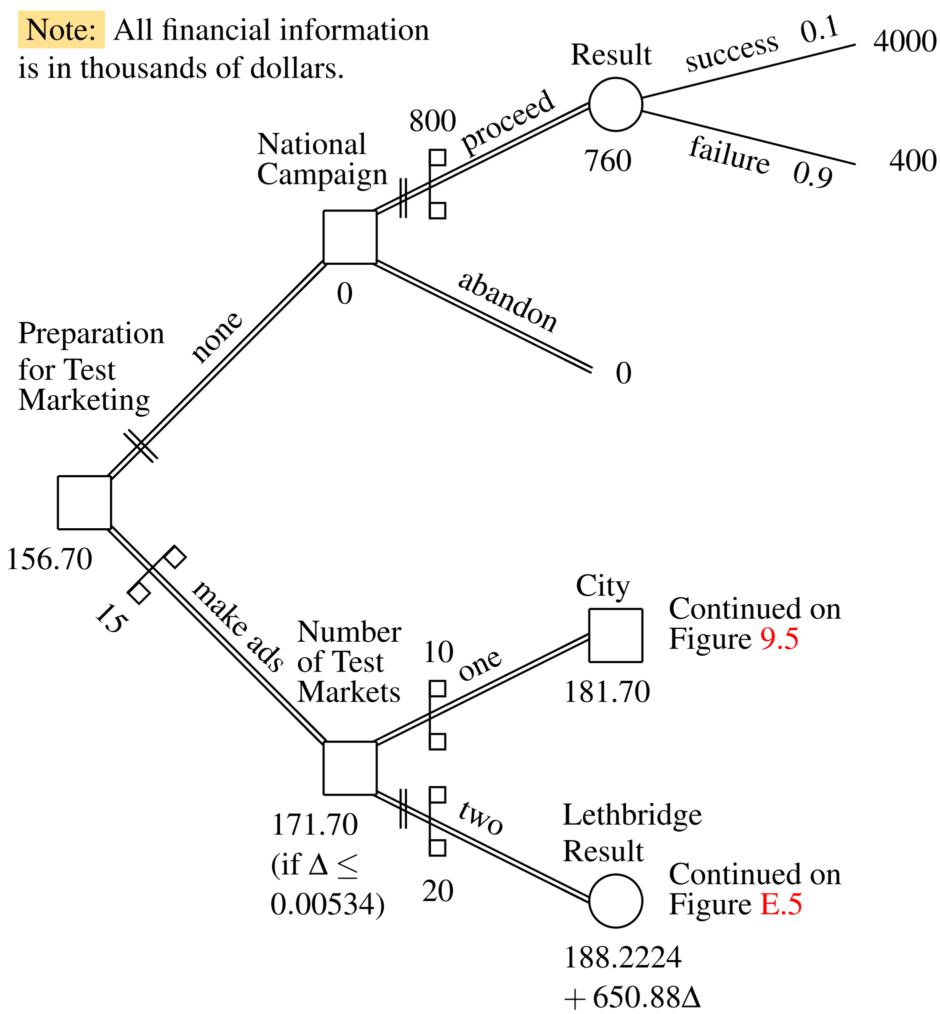


Figure E.6: Sensitivity Analysis – Beginning of the Tree

E.3.3 Sensitivity Problem

This is an extension to the Crop Planting problem found on page 450. It requires that the solution to the problem has been already found.

(a) Now suppose that the cost of planting potatoes is $\$60,000+\Delta$. For Δ both positive and negative, find the limits for which the recommendation from the original solution does not change.

(b) Now suppose that the cost of planting potatoes is fixed at \$60,000, but the probability of a mild blight on the first planting is $0.2+\Delta$, with the probability of a severe blight on the first planting being unchanged. For Δ both positive and negative, find the limits for which the recommendation from the original solution does not change.

(To avoid confusion between parts (a) and (b)), the terms Δc and Δp could be used.)